

Section 1.5 – Equations

- Linear equation المعادلة الخطية is an equation in the form

$$ax + b = 0$$

where a and b are real numbers and x is the variable.

لاحظ

Linear equations

$$4x - 5 = 3$$

$$2x = \frac{1}{2}x - 7$$

$$x - 6 = \frac{x}{3}$$

Nonlinear equations

$$x^2 + 2x = 8$$

$$\sqrt{x} - 6x = 0$$

$$\frac{3}{x} - 2x = 1$$

Example 1

Solve the equation $7x - 4 = 3x + 8$

Solution

$$7x - 3x = 8 + 4$$

$$4x = 12$$

$$x = \frac{12}{4}$$

$$x = 3$$

Example 2

The surface area A of a closed rectangular box can be calculated from its length l , width w , and height h according to the formula

$$A = 2lw + 2wh + 2lh$$

Solve for w in terms of the other variables in this equation.

Solution

$$\begin{aligned} A - 2lh &= 2lw + 2wh \\ A - 2lh &= w(2l + 2h) \\ \frac{A - 2lh}{2l + 2h} &= w \end{aligned}$$

- Quadratic equations المعادلات التربيعية are second-degree equations of the form

$$ax^2 + bx + c = 0$$

where a , b , and c are real numbers and $x \neq 0$

Example 3

Find all real solutions of the equation $x^2 + 5x = 24$

Solution

$$\begin{aligned} x^2 + 5x - 24 &= 0 \\ (x + 8)(x - 3) &= 0 \\ x + 8 = 0 \quad \text{or} \quad x - 3 &= 0 \\ x = -8 \quad \quad \quad x &= 3 \end{aligned}$$

Example 4

Find all real solutions of each equation.

(a) $x^2 = 5$

(b) $(x - 4)^2 = 5$

Solution

$$(a) \quad x^2 = 5$$

$$x = \pm\sqrt{5}$$

$$(b) \quad (x - 4)^2 = 5$$

$$x - 4 = \pm\sqrt{5}$$

$$x = 4 + \sqrt{5}$$

$$x = 4 - \sqrt{5}$$

- To make $x^2 + bx$ a perfect square (completing the square), add $\left(\frac{b}{2}\right)^2$

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Example 5

Find all real solutions of the equation $x^2 - 8x + 13 = 0$

Solution

$$x^2 - 8x + \left(-\frac{8}{2}\right)^2 = -13 + \left(-\frac{8}{2}\right)^2$$

$$x^2 - 8x + 16 = -13 + 16$$

$$(x - 4)^2 = 3$$

$$x - 4 = \pm\sqrt{3}$$

$$x = 4 + \sqrt{3}$$

$$x = 4 - \sqrt{3}$$

- The quadratic formula القانون العام can be used to find the roots (الحلول) of any quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

الجزء $b^2 - 4ac$ يسمى المميز discriminant

If $D > 0$, then the equation has **two** distinct real solutions.

If $D = 0$, then the equation has exactly one real solution.

If $D < 0$, then the equation has no real solution.

Example 6

Find all real solutions of each equation.

(a) $3x^2 - 5x - 1 = 0$

(b) $4x^2 + 12x + 9 = 0$

(c) $x^2 + 2x + 2 = 0$

Solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$(c) \quad x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$$

$$= \frac{-2 \pm \sqrt{4 - 8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} < 0$$

No real solution

Example 7

Solve the equation $\frac{3}{x} - \frac{2}{x-3} = \frac{-12}{x^2-9}$

Solution

$$\frac{3(x-3) - 2x}{x(x-3)} = \frac{-12}{(x-3)(x+3)}$$

$$\frac{(3x-9-2x)(\cancel{x-3})(x+3)}{\cancel{x-3}} = -12x$$

$$(x-9)(x+3) = -12x$$

$$x^2 - 9x + 3x - 27 = -12x$$

$$x^2 - 6x - 27 + 12x = 0$$

$$x^2 + 6x - 27 = 0$$

$$(x+9)(x-3) = 0$$

$$x = -9 \quad \text{or} \quad x = 3 \quad \text{X rejected}$$

Not part of
the domain

Check:

$$\text{LHS} = \frac{3}{-9} - \frac{2}{-9-3}$$

$$= -\frac{1}{3} - \frac{2}{-12} = \frac{-4+2}{12} = \frac{-2}{12} = \boxed{-\frac{1}{6}}$$

$$\text{RHS} = \frac{-12}{(-9)^2-9}$$

$$= \frac{-12}{72} = \boxed{-\frac{1}{6}}$$

$$\text{LHS} = \text{RHS}$$

$x = -9$
is a solution

Example 8

Solve the equation $2x = 1 - \sqrt{2-x}$ **Solution**

$$2x - 1 = -\sqrt{2-x}$$

$$(2x-1)^2 = 2-x$$

$$4x^2 - 4x + 1 = 2 - x$$

$$4x^2 - 4x + 1 - 2 + x = 0$$

$$4x^2 - 3x - 1 = 0$$

$$(4x+1)(x-1) = 0$$

$$4x+1=0$$

$$x = -\frac{1}{4}$$

$$x-1=0$$

$$x=1$$

Check:

$$\text{LHS} = 2\left(-\frac{1}{4}\right)$$

$$= -\frac{2}{4} = \boxed{-\frac{1}{2}}$$

$$\text{RHS} = 1 - \sqrt{2 - \left(-\frac{1}{4}\right)}$$

$$= 1 - \sqrt{2 + \frac{1}{4}}$$

$$= 1 - \sqrt{\frac{8+1}{4}}$$

$$= 1 - \sqrt{\frac{9}{4}} = 1 - \frac{3}{2}$$

$$= \boxed{-\frac{1}{2}}$$

$$\text{LHS} = 2(1) = 2$$

$$\text{RHS} = 1 - \sqrt{2-1} = 0$$

$$x=1$$

not a solution

Example 9

Find all solutions of the equation $x^4 - 8x^2 + 8 = 0$ **Solution**

$$(x^2)^2 - 8x^2 + 8 = 0$$

$$x^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 1 \cdot 8}}{2 \cdot 1}$$

$$= \frac{8 \pm \sqrt{64 - 32}}{2}$$

$$= \frac{8 \pm \sqrt{32}}{2} = 4 \pm \frac{\sqrt{16 \cdot 2}}{2}$$

$$= 4 \pm \frac{4}{2} \sqrt{2} = 4 \pm 2\sqrt{2}$$

$$x = \pm \sqrt{4 \pm 2\sqrt{2}}$$

$$x = \sqrt{4 + 2\sqrt{2}}$$

$$x = -\sqrt{4 + 2\sqrt{2}}$$

$$x = \sqrt{4 - 2\sqrt{2}}$$

$$x = -\sqrt{4 - 2\sqrt{2}}$$

Example 10

Find all solutions of the equation $x^{1/3} + x^{1/6} - 2 = 0$

Solution

$$(x^{1/6})^2 + x^{1/6} - 2 = 0$$

$$(x^{1/6} + 2)(x^{1/6} - 1) = 0$$

$$x^{1/6} = -2$$

$$x^{1/6} = 1$$

$$x = (-2)^6 = 64$$

$$x = 1^6 = 1$$

Check:

$$(64)^{1/3} + (64)^{1/6} - 2 = 4 \neq 0$$

 $x = 64$ is not a solution

$$1^{1/3} + 1^{1/6} - 2 = 0 \checkmark$$

 $x = 1$ is a solution

Example 11

Solve the equation $|2x - 5| = 3$

Solution

$$2x - 5 = 3$$

$$2x = 3 + 5$$

$$x = \frac{8}{2} = 4$$

$$2x - 5 = -3$$

$$2x = -3 + 5$$

$$x = \frac{2}{2} = 1$$

Problems

- Solve the linear equation $-x + 3 = 4x$.

$$-x - 4x = -3$$

$$-5x = -3$$

$$x = \frac{-3}{-5} = \frac{3}{5}$$

- Solve the equation for the variable R ; $PV = nRT$.

$$\frac{PV}{nT} = \frac{\cancel{P}R\cancel{T}}{\cancel{n}T}$$

$$R = \frac{PV}{nT}$$

- Find all real solutions of the equations by factoring.

(a) $x^2 + x - 12 = 0$

$$(x + 4)(x - 3) = 0$$

$$x + 4 = 0$$

$$x = -4$$

or

$$x - 3 = 0$$

$$x = 3$$

(b) $2x^2 = 8$

$$x^2 = \frac{8}{2} = 4$$

$$x = \pm \sqrt{4}$$

$$x = \pm 2$$

(c) $(2x - 5)^2 = 81$

$$2x - 5 = \pm \sqrt{81} = \pm 9$$

$$2x - 5 = 9$$

$$2x = 9 + 5 = 14$$

$$x = \frac{14}{2} = 7$$

or

$$2x - 5 = -9$$

$$2x = -9 + 5 = -4$$

$$x = \frac{-4}{2} = -2$$

- Find all real solutions of the equation by completing the square.

(a) $x^2 + 2x - 5 = 0$

$$x^2 + 2x + \left(\frac{2}{2}\right)^2 = 5 + \left(\frac{2}{2}\right)^2$$

$$x^2 + 2x + 1 = 5 + 1$$

$$(x + 1)^2 = 6$$

$$x + 1 = \pm\sqrt{6}$$

$$x = -1 + \sqrt{6} \quad \text{or} \quad x = -1 - \sqrt{6}$$

(b) $2x^2 + 8x + 1 = 0$

$$2x^2 + 8x = -1$$

$$2(x^2 + 4x) = -1$$

$$x^2 + 4x = -\frac{1}{2}$$

$$x^2 + 4x + \left(\frac{4}{2}\right)^2 = -\frac{1}{2} + \left(\frac{4}{2}\right)^2$$

$$x^2 + 4x + 4 = -\frac{1}{2} + 4$$

$$(x + 2)^2 = \frac{7}{2}$$

$$x + 2 = \pm\sqrt{\frac{7}{2}}$$

$$x = -2 + \sqrt{\frac{7}{2}} \quad \text{or} \quad x = -2 - \sqrt{\frac{7}{2}}$$

- Find all real solutions of the quadratic equation.

(a) $x^2 - 13x + 42 = 0$

$$(x-7)(x-6) = 0$$

$$x-7 = 0$$

$$x = 7$$

$$\text{or } x-6 = 0$$

$$x = 6$$

(b) $9x^2 + 12x + 4 = 0$

$$(3x+2)^2 = 0$$

$$3x+2 = 0$$

$$3x = -2$$

$$x = -\frac{2}{3}$$

perfect square

$$3x^2 + 3x \cdot 2 + 2^2$$

$$A^2 + 2AB + B^2 = (A+B)^2$$

- Find all real solutions of the equation.

$$(a) \frac{1}{x-1} + \frac{1}{x+2} = \frac{5}{4}$$

$$\frac{1(x+2) + 1(x-1)}{(x-1)(x+2)} = \frac{5}{4}$$

$$4(x+2 + x-1) = 5(x-1)(x+2)$$

$$4(2x+1) = 5(x^2 + x - 2)$$

$$8x + 4 = 5x^2 + 5x - 10$$

$$0 = 5x^2 + 5x - 8x - 10 - 4$$

$$5x^2 - 3x - 14 = 0$$

$$(5x+7)(x-2) = 0$$

$$5x+7=0$$

$$\text{or } x-2=0$$

$$x = -\frac{7}{5}$$

$$x = 2$$

Check

$$\begin{aligned} \text{LHS} &= \frac{1}{-\frac{7}{5}-1} + \frac{1}{-\frac{7}{5}+2} \\ &= \frac{1}{\frac{-7-5}{5}} + \frac{1}{\frac{-7+10}{5}} \\ &= \frac{5}{-12} + \frac{5}{3} = \frac{5}{4} \end{aligned}$$

$$\text{LHS} = \frac{1}{2-1} + \frac{1}{2+2}$$

$$= \frac{1}{1} + \frac{1}{4}$$

$$= \frac{5}{4} = \text{RHS}$$

$\therefore x=2$ is a solution

$\therefore x = -\frac{7}{5}$ is a solution

(b) $\sqrt{2x+1} + 1 = x$

$$\sqrt{2x+1} = x-1$$

$$2x+1 = (x-1)^2$$

$$2x+1 = x^2 - 2x + 1$$

$$0 = x^2 - 2x - 2x + \cancel{1} - \cancel{1}$$

$$x^2 - 4x = 0$$

$$x(x-4) = 0$$

$$x = 0$$

or

$$x - 4 = 0$$

$$x = 4$$

Check:

$$\text{LHS} = \sqrt{2(0)+1} + 1$$

$$= \sqrt{1} + 1 = 2$$

$$\text{RHS} = 0 \neq \text{LHS}$$

$x=0$ not a solution

$$\text{LHS} = \sqrt{2(4)+1} + 1$$

$$= \sqrt{8+1} + 1$$

$$= \sqrt{9} + 1 = 4$$

$$\text{RHS} = 4$$

$$\text{LHS} = \text{RHS}$$

$\therefore x=4$ is a solution

$$(c) x^4 - 13x^2 + 40 = 0$$

$$(x^2)^2 - 13x^2 + 40 = 0 \quad x^2 = W$$

$$W^2 - 13W + 40 = 0$$

$$(W - 8)(W - 5) = 0$$

$$W - 8 = 0$$

$$\text{or } W - 5 = 0$$

$$W = 8$$

$$W = 5$$

$$x^2 = 8$$

$$x^2 = 5$$

$$x = \pm\sqrt{8}$$

$$x = \pm\sqrt{5}$$

$$x = \sqrt{8} \text{ or } -\sqrt{8} \text{ or } \sqrt{5} \text{ or } -\sqrt{5}$$

$$(d) x^{4/3} - 5x^{2/3} + 6 = 0$$

$$(x^{2/3})^2 - 5x^{2/3} + 6 = 0 \quad x^{2/3} = W$$

$$W^2 - 5W + 6 = 0$$

$$(W - 3)(W - 2) = 0$$

$$W = 3$$

or

$$W = 2$$

$$x^{2/3} = 3$$

$$x^{2/3} = 2$$

$$x = 3^{3/2}$$

$$x = 2^{3/2}$$

$$x = \sqrt{27}$$

$$x = \sqrt{8}$$

By checking both values $\left\{ \begin{array}{l} x = \sqrt{27} \checkmark \checkmark \\ x = \sqrt{8} \checkmark \checkmark \end{array} \right.$

(e) $|3x + 5| = 1$

$$3x + 5 = 1$$

$$3x = 1 - 5$$

$$x = \frac{-4}{3}$$

or $3x - 5 = -1$

$$3x = -1 - 5$$

$$x = \frac{-6}{3}$$

$$x = -2$$

$$\text{S.S.} \therefore \left\{ -2, -\frac{4}{3} \right\}$$