Section 1.2 – Exponents and Radicals

- A product of identical numbers is written in exponential notation

حاصل ضرب رقم في نفسه يكتب على شكل أساس و أس

(عدد مرات التكرار) Exponent الأس

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}$$

Base الأساس

Example 1

Find the values of the following exponents.

(a) $\left(\frac{1}{2}\right)^5$

(b) $(-3)^4$

(c) -3^4

Solution

(a) $\left(\frac{1}{2}\right)^5 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$

(b) $(-3)^4 = (-3)(-3)(-3)(-3) = 81$

(c) $-3^4 = -(3.3.3.3) = -81$

لاحظ

If $a \neq 0$ is a real number and n is a positive integer, then

$a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$

Example 2

Find the values of the following exponents.

(a) $\left(\frac{4}{7}\right)^0$

(b) $(-2)^{-3}$

Solution

(a) $\left(\frac{4}{7}\right)^0 = 1$

(b) $(-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{-8}$

- Laws of exponents قوانين الأسس

1. $a^m a^n = a^{m+n}$ $\quad 3^2 \cdot 3^5 = 3^{2+5} = 3^7$

2. $\frac{a^m}{a^n} = a^{m-n}$ $\quad \frac{3^5}{3^2} = 3^{5-2} = 3^3$

3. $(a^m)^n = a^{mn}$ $\quad (3^2)^5 = 3^{2 \cdot 5} = 3^{10}$

4. $(ab)^n = a^n b^n$ $\quad (3 \cdot 4)^2 = 3^2 4^2$

5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $\quad \left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2}$

6. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ $\quad \left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$

7. $\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$ $\quad \frac{3^{-2}}{4^{-5}} = \frac{4^5}{3^2}$

Example 3

Simplify.

(a) x^4x^7

(b) y^4y^{-7}

(c) $\frac{c^9}{c^5}$

(d) $(b^4)^5$

(e) $(3x)^3$

(f) $\left(\frac{x}{2}\right)^5$

Solution

(a) $x^4 x^7 = x^{4+7} = x^{11}$

(b) $y^4 y^{-7} = y^{4-7} = y^{-3} = \frac{1}{y^3}$

(c) $\frac{c^9}{c^5} = c^{9-5} = c^4$

(d) $(b^4)^5 = b^{4.5} = b^{20}$

(e) $(3x)^3 = 3^3 x^3 = 27x^3$

(f) $\left(\frac{x}{2}\right)^5 = \frac{x^5}{2^5} = \frac{x^5}{32}$

Example 4

Simplify $(2a^3b^2)(3ab^4)^3$

Solution

$2a^3b^2(3^3a^3b^{4.3})$

$= 2a^3b^2(27a^3b^{12})$

$= 2.27\,a^{3+3}b^{2+12}$

$= 54a^6b^{14}$

- If n is a <u>positive integer</u>

مهم جدا: إذا كان n
عدد زوجي، لازم:
$a \geq 0$ and $b \geq 0$

$\sqrt[n]{a} = b$ means $b^n = a$

مثال

$\sqrt[4]{81} = 3$ because $3^4 = 81$

لاحظ

$\sqrt{a} = b$ means $b^2 = a$

جذر تربيعي "square root" وقيمة $n = 2$ و لا نحتاج أن نكتبها

- Properties of nth roots خواص الجذور

1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ $\sqrt[3]{-8 \cdot 27} = \sqrt[3]{-8}\sqrt[3]{27} = (-2)(3) = -6$

2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $\sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{2}{3}$

3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$ $\sqrt{\sqrt[3]{729}} = \sqrt[6]{729} = 3$

4. $\sqrt[n]{a^n} = a$ if n is odd $\sqrt[3]{(-5)^3} = -5$, $\sqrt[5]{2^5} = 2$

5. $\sqrt[n]{a^n} = |a|$ if n is even $\sqrt[4]{(-5)^4} = |-5| = 5$

Example 5

Simplify $\sqrt[4]{81x^8y^4}$

Solution

$\sqrt[4]{81}\ \sqrt[4]{x^8}\ \sqrt[4]{y^4}$

$= 3x^2y$

لاحظ

$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

Example 6

Simplify $\sqrt{32} + \sqrt{200}$

Solution

$\sqrt{16.2} + \sqrt{100.2}$

$= \sqrt{16}\sqrt{2} + \sqrt{100}\sqrt{2}$

$= 4\sqrt{2} + 10\sqrt{2} = 14\sqrt{2}$

- Rational exponents

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

where m and n are integers and $n > 0$

Note: If n is even عدد زوجي then a must be ≥ 0

Example 7

Solve.

(a) $4^{1/2}$

(b) $8^{-2/3}$

Solution

(a) $4^{1/2} = \sqrt[2]{4} = \pm 2$

(b) $8^{-2/3} = \frac{1}{8^{2/3}} = \frac{1}{(8^{1/3})^2} = \frac{1}{2^2} = \frac{1}{4}$

- Rationalizing the denominator عملية نقوم بيها للتخلص من الجذور في المقام

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$$

تساوي 1

$$\frac{1}{\sqrt[n]{a^m}} = \frac{1}{\sqrt[n]{a^m}} \cdot \frac{\sqrt[n]{a^{n-m}}}{\sqrt[n]{a^{n-m}}} = \frac{\sqrt[n]{a^{n-m}}}{a}$$

Example 8

Put each fractional expression into standard form by rationalizing the denominator.

(a) $\frac{2}{\sqrt{3}}$

(b) $\frac{1}{\sqrt[3]{5}}$

(c) $\sqrt[7]{\frac{1}{a^2}}$

Solution

(a) $\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

(b) $\frac{1}{\sqrt[3]{5}} \cdot \frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}} = \frac{\sqrt[3]{25}}{5}$

(c) $\sqrt[7]{\frac{1}{a^2}} = \frac{\sqrt[7]{1}}{\sqrt[7]{a^2}} = \frac{1}{\sqrt[7]{a^2}}$

$= \frac{1}{\sqrt[7]{a^2}} \cdot \frac{\sqrt[7]{a^5}}{\sqrt[7]{a^5}}$

$= \frac{\sqrt[7]{a^5}}{a}$

Problems

- Evaluate the expression -2^6

$-(2.2.2.2.2.2)$

$= -64$

- Simplify, and eliminate any negative exponents.

(a) $x^{-5} \cdot x^3$

$x^{-5+3} = x^{-2} = \frac{1}{x^2}$

(b) $w^{-2}w^{-4}w^5$

w^{-2-4+5}

$= w^{-1} = \frac{1}{w}$

(c) $\frac{x^{16}}{x^{10}}$

$= x^{16-10} = x^6$

- Simplify, and eliminate any negative exponents.

(a) $(3x^3y^2)(2y^3)$

$= 3.2\ x^3\ y^{2+3}$

$= 6x^3y^5$

(b) $(5w^2z^{-2})^2(z^3)$

$= (5^2\ w^{2.2}\ z^{-2.2})(z^3)$

$= 25\ w^4\ z^{-4+3}$

$= 25\ w^4\ z^{-1}$

$= \dfrac{25w^4}{z}$

(c) $\frac{a^9a^{-2}}{a}$

$= a^{9-2-1}$

$= a^6$

- Simplify the expression.

(a) $\sqrt[6]{64a^6b^7}$

$= \sqrt[6]{64}\ \sqrt[6]{a^6}\ \sqrt[6]{b^7}$

$= 2a\sqrt[6]{b^7}$

(b) $\sqrt[3]{a^2b}\ \sqrt[3]{64a^4b}$

$= \sqrt[3]{64\ a^2\ a^4\ b\ b}$

$= 4\sqrt[3]{a^6\ b^2}$

$= 4a^2\sqrt[3]{b^2}$

(c) $\sqrt{32} + \sqrt{18}$

$= \sqrt{4.8} + \sqrt{9.2}$

$= 2\sqrt{8} + 3\sqrt{2}$

(d) $\sqrt{81x^2 + 81}$

$= \sqrt{81(x^2+1)}$

$= 9\sqrt{x^2+1}$

- Evaluate each expression.

(a) $32^{2/5}$

$= \left(32^{1/5}\right)^2$

$= 2^2 = 4$

(b) $\left(\frac{4}{9}\right)^{-1/2}$

$= \left(\frac{9}{4}\right)^{1/2}$

$= \sqrt{\frac{9}{4}} = \frac{3}{2}$

(c) $\left(\frac{16}{81}\right)^{3/4}$

$= \left(\left(\frac{16}{81}\right)^{1/4}\right)^3$

$= \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

- Simplify the expression, and eliminate any negative exponent(s).

(a) $\frac{(8s^3t^3)^{2/3}}{(s^4t^{-8})^{1/4}}$

$$= \frac{\left(8^{1/3}\ s^{3\cdot 1/3}\ t^{3\cdot 1/3}\right)^2}{s^{4\cdot\frac{1}{4}}\ t^{-8\cdot\frac{1}{4}}}$$

$$= \frac{(2\ s\ t)^2}{s\ t^{-2}} =$$

$$= \frac{4\ s^{\cancel{2}} t^2 \cdot t^2}{\cancel{s}} = 4\ s\ t^4$$

(b) $\left(\frac{x^{3/2}}{y^{-1/2}}\right)^4 \left(\frac{x^{-2}}{y^3}\right)$

$$= x^{\frac{3}{2}\cdot 4}\ \ y^{\frac{1}{2}\cdot 4} \left(\frac{1}{y^3 x^2}\right)$$

$$= \frac{x^6\ y^2}{y^3\ x^2} = \frac{x^4}{y}$$

- Simplify the expression, and eliminate any negative exponent(s).

(a) $\sqrt[6]{y^5}\,\sqrt[3]{y^2}$

$= y^{5/6}\; y^{2/3}$

$= y^{5/6 + 2/3}$

$= y^{\frac{5+4}{6}} = y^{9/6}$

$= y^{3/2} = \sqrt[3]{y^2}$

(b) $\sqrt[6]{y\sqrt{y}}$

$= y^{\frac{1}{6}}\; y^{\frac{1}{2}\cdot\frac{1}{6}}$

$= y^{\frac{1}{6}}\; y^{\frac{1}{12}}$

$= y^{\frac{2+1}{12}} = y^{3/12}$

$= y^{1/4} = \sqrt[4]{y}$

- Put each fractional expression into standard form by rationalizing the denominator.

(a) $\frac{1}{\sqrt{6}}$

$$= \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

(b) $\sqrt{\frac{3}{2}}$

$$= \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{3.2}}{2} = \frac{\sqrt{6}}{2}$$

(c) $\frac{9}{\sqrt[4]{2}}$

$$= \frac{9}{\sqrt[4]{2}} \cdot \frac{\sqrt[4]{2^3}}{\sqrt[4]{2^3}}$$

$$= \frac{9\sqrt[4]{8}}{2}$$