Section 3.3 – Dividing Polynomials

- If $P(x)$ and $D(x)$ are polynomials دوال كثيرة حدود, and $D(x) \neq 0$, then we can divide نقسم $P(x)$ by $D(x)$ such that:

Dividend, Quotient, Remainder, Divisor

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)} \quad \Longrightarrow \quad P(x) = D(x) \cdot Q(x) + R(x)$$

يمكن قسمة دالة كثيرة حدود على دالة كثيرة حدود بنفس طريق قسمة رقم على رقم، و تسمى هذه الطريقة:

القسمة المطولة long division

لاحظ

طريقة قسمة الدوال نفس الأرقام

Dividend, Remainder, Divisor, Quotient

$$\frac{38}{7} = 5 + \frac{3}{7}$$

Example 1

Divide $6x^2 - 26x + 12$ by $x - 4$

Solution

$$\begin{array}{r} 6x - 2 \\ x-4 \overline{)\,6x^2 - 26x + 12} \\ -6x^2 \pm 24x \\ \hline -2x + 12 \\ +\,-2x\,-8 \\ \hline 4 \end{array}$$

$$\frac{6x^2-26x+12}{x-4} = 6x - 2 + \frac{4}{x-4}$$

Example 2

Let $P(x) = 8x^4 + 6x^2 - 3x + 1$ and $D(x) = 2x^2 - x + 2$. Divide $P(x)$ by $D(x)$.

Solution

$$2x^2 - x + 2 \;\overline{)\; 8x^4 + 6x^2 - 3x + 1}$$

$4x^2 + 2x$

$-(8x^4 - 4x^3 + 8x^2)$

$4x^3 - 2x^2 - 3x + 1$

$-(4x^3 - 2x^2 + 4x)$

$-7x + 1$

$$\frac{8x^4 + 6x^2 - 3x + 1}{2x^2 - x + 2} = 4x^2 + 2x + \frac{-7x + 1}{2x^2 - x + 2}$$

- Synthetic division القسمة التركيبية

نقدر نستخدمها فقط إذا كان المقسوم عليه على صورة $x - c$ (جمع أو طرح)

Example 3

Use synthetic division to divide $2x^3 - 7x^2 + 5$ by $x - 3$

Solution

Synthetic division

3	2	−7	0	5
		6	−3	−9
	2	−1	−3	−4

Long division

$$2x^2 - x - 3$$
$$x-3 \overline{)\, 2x^3 - 7x^2 + 5}$$
$$-2x^3 + 6x^2$$
$$-x^2 + 5$$
$$+x^2 - 3x$$
$$-3x + 5$$
$$-3x + 9$$
$$-4$$

$$\frac{2x^3 - 7x^2 + 5}{x - 3} = 2x^2 - x - 3 - \frac{4}{x - 3}$$

- **Remainder Theorem:**

إذا كان المقسوم عليه على صورة $x - c$ فإن متبقي القسمة يساوي $\boldsymbol{P(c)}$

Example 4

Find the remainder of dividing $P(x) = x^3 + 3x^2 - 7x + 6$ by $x - 2$

Solution

$$\text{remainder} = p(2) = 2^3 + 3(2)^2 - 7 \cdot 2 + 6$$

$$= 8 + 12 - 14 + 6$$

$$= 12$$

Another sol.

$$x - 2 \overline{)\, x^3 + 3x^2 - 7x + 6}$$

$$x^2 + 5x + 3$$

$$-x^3 \pm 2x^2$$

$$5x^2 - 7x + 6$$

$$-5x^2 \pm 10x$$

$$-3x + 6$$

$$-3x \pm 6$$

$$12$$

Example 5

Find the remainder of dividing $P(x) = x^3 + 3x^2 - 7x + 6$ by $x + 2$

Solution

$$p(-2) = (-2)^3 + 3(-2)^2 - 7(-2) + 6$$

$$= -8 + 12 + 14 + 6$$

$$= 24$$

- **Factor Theorem:** c is a zero of P if, and only if, $x - c$ is a factor of $P(x)$

إذا كانت $P(c) = 0$ فإن c هي أحد أصفار الدالة $P(x)$ وبالتالي فإن $x - c$ هي أحد معاملات (factors) $P(x)$

أي أن $P(x)$ تقبل القسمة على $x - c$ بدون متبقي remainder

Example 6

Which of the following is a factor of $P(x) = x^3 - 7x - 6$?

(a) $x + 2$ (b) $x - 1$ (c) $x + 4$ (d) $x + 3$

Solution

(a) $P(-2) = (-2)^3 - 7(-2) - 6 = -8 + 14 - 6 = 0$

(b) $P(1) = 1^3 - 7 \cdot 1 - 6 = -12$

(c) $P(-4) = (-4)^3 - 7(-4) - 6 = -42$

(d) $P(-3) = (-3)^3 - 7(-3) - 6 = -12$

Example 7

Let $P(x) = x^3 - 7x + 6$. Show that 1 is a zero of $P(x)$, and use this fact to find all other real zeros.

Solution

$P(1) = 1^3 - 7(1) + 6 = 0$

$\therefore$ 1 is a zero of $P(x)$

1	1	0	−7	6
	↓	1	1	−6
	1	1	−6	0

$P(x) = (x - 1)(x^2 + x - 6)$

$= (x - 1)(x + 3)(x - 2)$

zeros of $P(x)$ are: $\{1, -3, 2\}$

Problems

- Two polynomials P and D are given. Use either synthetic or long division to divide $P(x)$ by $D(x)$

(a) $P(x) = -x^3 - 2x + 6, D(x) = x + 1$

$$\begin{array}{r|cccc} -1 & -1 & 0 & -2 & 6 \\ & & 1 & -1 & 3 \\ \hline & -1 & 1 & -3 & 9 \end{array}$$

$$\frac{-x^3 - 2x + 6}{x+1} = -x^2 + x - 3 + \frac{9}{x+1}$$

(b) $P(x) = 8x^4 + 4x^3 + 6x^2, D(x) = 2x^2 + 1$

$$\begin{array}{r} 4x^2 + 2x + 1 \\ 2x^2+1 \overline{)\,8x^4 + 4x^3 + 6x^2} \\ \underline{-8x^4 \qquad -4x^2} \\ 4x^3 + 2x^2 \\ \underline{-4x^3 \qquad -2x} \\ 2x^2 - 2x \\ \underline{-2x^2 \qquad -1} \\ -2x - 1 \end{array}$$

$$\frac{8x^4 + 4x^3 + 6x^2}{2x^2+1} = 4x^2 + 2x + 1 + \frac{-2x-1}{2x^2+1}$$

- Find the quotient and remainder using long division

$$\frac{x^3 + 2x + 1}{x^2 - x + 3}$$

$$\begin{array}{r} x + 1 \\ x^2 - x + 3 \overline{)\, x^3 \quad\quad + 2x + 1} \\ \underline{-x^3 + x^2 - 3x} \\ x^2 - x + 1 \\ \underline{-x^2 + x - 3} \\ -2 \end{array}$$

Quotient $= x + 1$

Remainder $= -2$

- Find the quotient and remainder using synthetic division

$$\frac{x^3 - 8x + 2}{x + 3}$$

$$\begin{array}{r|cccc} -3 & 1 & 0 & -8 & 2 \\ & & -3 & 9 & -3 \\ \hline & 1 & -3 & 1 & -1 \end{array}$$

Quotient $= x^2 - 3x + 1$

Remainder $= -1$

- Use synthetic division and the Remainder Theorem to evaluate $P(c)$

$P(x) = 4x^2 + 12x + 5, c = -1$

$$\begin{array}{r|rrr} -1 & 4 & 12 & 5 \\ & & -4 & -8 \\ \hline & 4 & 8 & -3 \end{array}$$

$P(-1) = -3$

- Use the Factor Theorem to show that $x - c$ is a factor of $P(x)$ for the given value of c.

$P(x) = x^3 - 3x^2 + 3x - 1, c = 1$

$P(1) = 1^3 - 3(1)^2 + 3(1) - 1$

$= 1 - 3 + 3 - 1 = 0$

$\therefore P(1) = 0$

$\therefore x - 1$ is a factor of $P(x)$

-Show that the given value of c is a zero of $P(x)$, and find all other zeros of $P(x)$

$P(x) = x^3 + 2x^2 - 9x - 18, c = -2$

$$P(-2) = (-2)^3 + 2(-2)^2 - 9(-2) - 18$$
$$= -8 + 8 + 18 - 18 = 0$$

$\therefore$ -2 is a zero of $P(x)$

$$\begin{array}{r|cccc} -2 & 1 & 2 & -9 & -18 \\ & & -2 & 0 & 18 \\ \hline & 1 & 0 & -9 & 0 \end{array}$$

$$\therefore x^3 + 2x^2 - 9x - 18 = (x+2)(x^2-9)$$
$$= (x+2)(x-3)(x+3)$$

zeros: $\{-2, 3, -3\}$