Section 3.4 – Real Zeros of Polynomials

- **The Rational Zero Theorem**: For any polynomial function

Leading coefficient   Constant term

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

$$\text{Possible rational zeros} = \frac{\text{Factors of the constant term}}{\text{Factors of the leading coefficient}}$$

الحلول (الأصفار) المحتملة لأي دالة كثيرة الحدود يجب أن تكون ناتجة عن قسمة تحليل الحد الثابت (factors of the constant term) على تحليل معامل أول حد (factors of leading coefficient)

Example 1

List all possible rational zeros of $f(x) = -x^4 + 3x^2 + 4$

Solution

$$\frac{\pm 4, \pm 1, \pm 2}{\pm 1} = \pm 4, \pm 1, \pm 2$$

Example 2

List all possible rational zeros of $f(x) = 2x^3 + x^2 - 13x + 6$

Solution

$$\frac{\pm 6, \pm 1, \pm 3, \pm 2}{\pm 1, \pm 2} = \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$$

لاحظ

- نستطيع إيجاد حلول (أصفار) الدالة باتباع الخطوات التالية:

1- إيجاد الأصفار المحتملة

2- نستخدم القسمة التركيبية لاختبار الأصفار المحتملة من الخطوة السابقة

3- نعيد الخطوة السابقة حتى نصل إلى معادلة تربيعية ثم نقوم بتحليلها بالطرق التقليدية

Example 3

Given $P(x) = x^3 - 3x + 2$

(a) Find the set of all possible zeros of $P(x)$.

(b) Factor $P(x)$ completely.

(c) Find all zeros of $P(x)$.

Solution

(a) $\pm 1, \pm 2$

(b)

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & 2 \\ & \downarrow & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & 4 \end{array}$$

-1 is not a zero

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ & \downarrow & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

1 is a zero

$$P(x) = (x-1)(x^2 + x - 2)$$
$$= (x-1)(x+2)(x-1)$$

(c) zeros: $\{1, -2\}$

Example 4

Find all zeros of $P(x) = 3x^3 + 5x^2 - 2x - 4$

Solution

$$\frac{\pm 1, \pm 2, \pm 4}{\pm 1, \pm 3} = \pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$$

$$\begin{array}{r|rrrr} -1 & 3 & 5 & -2 & -4 \\ & \downarrow & -3 & -2 & 4 \\ \hline & 3 & 2 & -4 & 0 \end{array}$$

$P(x) = (x+1)(3x^2 + 2x - 4)$

$3x^2 + 2x - 4$ cannot be factored

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3(-4)}}{2 \cdot 3}$$

$$= \frac{-2 \pm \sqrt{52}}{6} = \frac{-2 \pm 2\sqrt{13}}{6}$$

$$= \frac{-1 \pm \sqrt{13}}{3}$$

zeros: $\left\{-1, \frac{-1+\sqrt{13}}{3}, \frac{-1-\sqrt{13}}{3}\right\}$

Problems

- Given $P(x) = 2x^3 + x^2 - 13x + 6$

(a) Factor $P(x)$ completely.

Potential zeros: $\dfrac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 2}$

: $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$

$P(1) = 2(1)^3 + 1^2 - 13(1) + 6 = -4$

$P(2) = 2(2)^3 + 2^2 - 13(2) + 6 = 0$

$$\begin{array}{r|rrrr} 2 & 2 & 1 & -13 & 6 \\ & & 4 & 10 & -6 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

$P(x) = (x-2)(2x^2+5x-3)$

$= (x-2)(2x-1)(x+3)$

(b) Find all zeros of $P(x)$.

Zeros: $\{2, \frac{1}{2}, -3\}$

- Let $P(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$. Find all zeros of $P(x)$.

potential zeros: $\pm 1, \pm 2, \pm 5, \pm 10$

$P(1) = 1^4 - 5(1)^3 - 5(1)^2 + 23(1) + 10 = 24$

$P(2) = 2^4 - 5(2)^3 - 5(2)^2 + 23(2) + 10 = 12$

$P(5) = 5^4 - 5(5)^3 - 5(5)^2 + 23(5) + 10 = 0$

5	1	-5	-5	23	10
		5	0	-25	-10
	1	0	-5	-2	0

$P(x) = (x-5)(x^3 - 5x - 2)$

$Q(x) = x^3 - 5x - 2$

Potential zeros: $\pm 1, \pm 2$

$Q(1) = 1 - 5(1) - 2 = -6$

$Q(-2) = (-2)^3 - 5(-2) - 2 = 0$

-2	1	0	-5	-2
		-2	4	2
	1	-2	-1	0

$Q(x) = (x+2)(x^2 - 2x - 1)$

$\therefore P(x) = (x-5)(x+2)(x^2 - 2x - 1)$

for $x^2 - 2x - 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot -1}}{2 \cdot 1}$$

$$= \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm 2\sqrt{2}}{2}$$

$$= 1 \pm \sqrt{2}$$

$\therefore$ Zeros: $\{5, -2, 1+\sqrt{2}, 1-\sqrt{2}\}$