Section 7.1 – Trigonometric Identities

- Trigonometric identities الاقترانات المثلثية are equations relating trigonometric functions of angles as follows:

Reciprocal identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos}{\sin x}$$

Pythagorean identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad \cot^2 x + 1 = \csc^2 x$$

Even-Odd identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

Cofunction identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$$

Example 1

Simplify the expression $\cos t + \tan t \sin t$.

Solution

$$\cos t + \frac{\sin t}{\cos t} \cdot \sin t$$

$$= \cos t + \frac{\sin^2 t}{\cos t} = \frac{\cos^2 t}{\cos t} + \frac{\sin^2 t}{\cos t}$$

$$= \frac{\cos^2 t + \sin^2 t}{\cos t} = \frac{1}{\cos t} = \sec t$$

Example 2

Simplify the expression.

(a) $\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{1+\sin\theta}$

(b) $\frac{1+\sin x}{\cos x}+\frac{\cos x}{1+\sin}$

Solution

(a) $$\frac{(\sin\theta)(1+\sin\theta)+\cos\theta\cos\theta}{(\cos\theta)(1+\sin\theta)}$$

$$=\frac{\sin\theta+\sin^2\theta+\cos^2\theta}{(\cos\theta)(1+\sin\theta)}$$

$$=\frac{\cancel{\sin\theta+1}}{(\cos\theta)\cancel{(1+\sin\theta)}}=\frac{1}{\cos\theta}=\sec\theta$$

(b) $$\frac{(1+\sin x)(1+\sin x)+\cos x\cos x}{(\cos x)(1+\sin x)}$$

$$=\frac{1+2\sin x+\sin^2 x+\cos^2 x}{(\cos x)(1+\sin x)}$$

$$=\frac{1+2\sin x+1}{(\cos x)(1+\sin x)}=\frac{2+2\sin x}{(\cos x)(1+\sin x)}$$

$$=\frac{2\cancel{(1+\sin x)}}{(\cos x)\cancel{(1+\sin x)}}=\frac{2}{\cos x}=2\sec x$$

Example 3

Verify (prove) the identity.

(a) $\frac{\cos\theta}{\sec\theta\sin\theta} = \csc\theta - \sin\theta$

(b) $\cos(-t) - \sin(-t) = \cos t + \sin t$

Solution

(a) $RHS = \csc\theta - \sin\theta$

$$= \frac{1}{\sin\theta} - \sin\theta = \frac{1-\sin^2\theta}{\sin\theta}$$

$$= \frac{\cos^2\theta}{\sin\theta}$$

$$LHS = \frac{\cos\theta}{\frac{1}{\cos\theta}\sin\theta} = \cos\theta \div \frac{\sin\theta}{\cos\theta}$$

$$= \cos\theta \cdot \frac{\cos\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta} = RHS$$

(b) $\cos(-t) = \cos t$

$\sin(-t) = -\sin t$

$\therefore \cos(-t) - \sin(-t)$

$= \cos t - (-\sin t)$

$= \cos t + \sin t \quad = RHS$

Problems

- Simplify the expression.

(a) $\tan^2 x - \sec^2 x$

$$\tan^2 x + 1 = \sec^2 x$$

$$\tan^2 x - \sec^2 x = -1$$

(b) $\frac{\sin x \sec x}{\tan x}$

$$\frac{\sin x \ \frac{1}{\cos x}}{\tan x}$$

$$= \frac{\frac{\sin x}{\cos x}}{\tan x}$$

$$= \frac{\tan x}{\tan x} = 1$$

- Verify (prove) the identity.

(a) $\cos\theta\,(\sec\theta - \cos\theta) = \sin^2\theta$

$$LHS = \cos\theta\left(\frac{1}{\cos\theta} - \cos\theta\right)$$

$$= \frac{\cos\theta}{\cos\theta} - \cos^2\theta$$

$$= 1 - \cos^2\theta = \sin^2\theta = RHS$$

(b) $\frac{\cos\theta}{1-\sin\theta} = \sec\theta + \tan\theta$

$$LHS = \frac{\cos\theta}{1-\sin\theta} \cdot \frac{1+\sin\theta}{1+\sin\theta}$$

$$= \frac{\cos\theta\,(1+\cos\theta)}{1-\sin^2\theta}$$

$$= \frac{\cancel{\cos\theta}\,(1+\sin\theta)}{\cos^{\cancel{2}}\theta}$$

$$= \frac{1+\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta + \tan\theta = RHS$$

(c) $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

$$LHS = (\sin\theta + \cos\theta)^2$$

$$= \sin^2\theta + 2\sin\theta\cos\theta + \sin^2\theta$$

$$= 1 + 2\sin\theta\cos\theta = RHS$$

(d) $\dfrac{\cos\theta}{\sec\theta} + \dfrac{\sin\theta}{\csc\theta} = 1$

$$LHS = \cos\theta \div \frac{1}{\cos\theta} + \sin\theta \div \frac{1}{\sin\theta}$$

$$= \cos\theta \cdot \cos\theta + \sin\theta \cdot \sin\theta$$

$$= \cos^2\theta + \sin^2\theta = 1 = RHS$$

(e) $\frac{\cot\theta}{\csc\theta-\sin\theta}=\sec\theta$

$$LHS = \cot\theta \div (\csc\theta - \sin\theta)$$

$$= \frac{\cos\theta}{\sin\theta} \div \left(\frac{1}{\sin\theta} - \sin\theta\right)$$

$$= \frac{\cos\theta}{\sin\theta} \div \left(\frac{1-\sin^2\theta}{\sin\theta}\right)$$

$$= \frac{\cos\theta}{\sin\theta} \div \frac{\cos^2\theta}{\sin\theta}$$

$$= \frac{\cancel{\cos\theta}}{\cancel{\sin\theta}} \cdot \frac{\cancel{\sin\theta}}{\cos^{\cancel{2}}\theta} = \frac{1}{\cos\theta} = \sec\theta$$

$$= RHS$$

(f) $\frac{1-\cos\theta}{\sin\theta}=\frac{\sin\theta}{1+\cos\theta}$

$$LHS = \frac{1-\cos\theta}{\sin\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta}$$

$$= \frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{\sin^{\cancel{2}}\theta}{\cancel{\sin\theta}(1+\cos\theta)}$$

$$= \frac{\sin\theta}{1+\cos\theta} = RHS$$