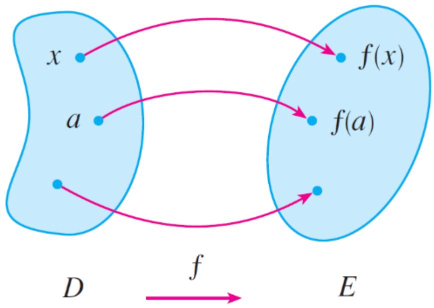
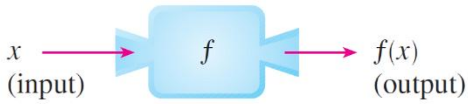
Section 1.2 – Mathematical Models: A Catalog of Essential Functions

- الدوال Functions علاقة بين مجموعة مدخلات وتسمى المجال domain، ومجموعة مخرجات تسمى المدى range

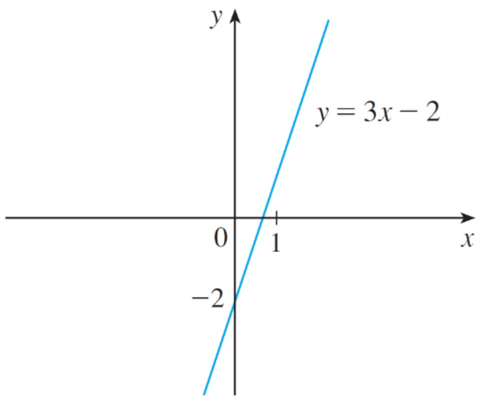




- Linear functions الدوال الخطية (1/8):

$$y = f(x) = mx + b$$

x	$y = f(x) = 3x - 2$
1.0	1.0
1.1	1.3
1.2	1.6
1.3	1.9
1.4	2.2
1.5	2.5



- Polynomials الحدوديات، أو كثيرة الحدود (2/8):

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

Domain: $\mathbb{R} = (-\infty, \infty)$

Example

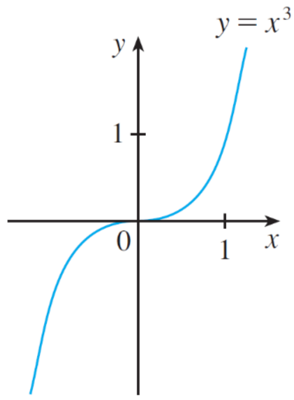
$$P(x) = 2x^6 - x^4 + \tfrac{2}{5}x^3 + \sqrt{2}$$

Degree = 6

Example

$$P(x) = ax^3 + bx^2 + cx + d, \qquad a \neq 0$$

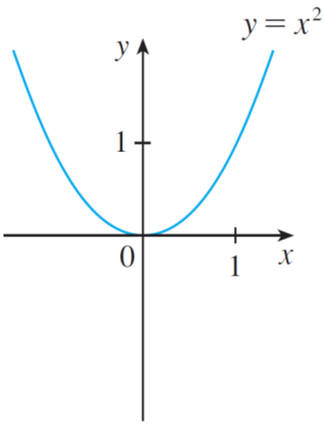
Degree = 3 , cubic



Example

$$P(x) = ax^2 + bx + c, \qquad a \neq 0$$

Degree = 2 , quadratic



تذكر

quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

القانون العام

- Power functions دوال القوى (3/8):

$$f(x) = x^a$$

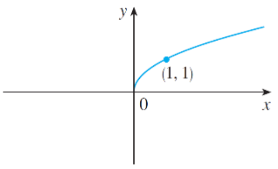
$a = n$ | $a = 1/n$ | $a = -1$

x^2 | $\sqrt[3]{x}$ | $\frac{1}{x}$

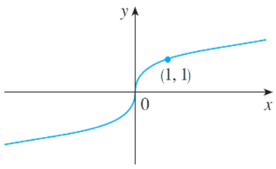
Domain: $\mathbb{R} = (-\infty, \infty)$

Domain:
n even ➔ $(0, \infty)$
n odd ➔ $\mathbb{R}$

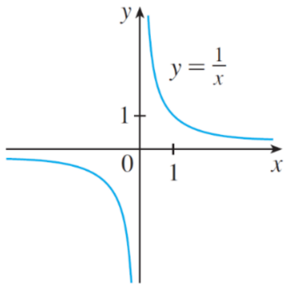
Domain: $\mathbb{R}/\{0\}$



(a) $f(x) = \sqrt{x}$



(b) $f(x) = \sqrt[3]{x}$



root function

reciprocal function

- Rational functions الدوال النسبية (4/8):

$$f(x) = \frac{P(x)}{Q(x)}$$

Domain: $\mathbb{R}/\{\text{zeros of the denominator}\}$

Example 1

Find the domain of the function $f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$

Solution

$x^2 - 4 = 0$

$(x-2)(x+2) = 0$

$x = 2 \quad x = -2$

$\therefore D_f : \mathbb{R} / \{-2, 2\}$

- Algebraic functions (5/8):

دالة تتكون من تركيب دالتين أو أكثر بالجمع addition، الطرح subtraction، الضرب multiplication، القسمة، والوضع تحت الجذور taking roots

$$f(x) = \sqrt{x^2+1} \qquad g(x) = \frac{x^4-16x^2}{x+\sqrt{x}} + (x-2)\sqrt[3]{x+1}$$

Example 2

Find the domain of the function.

(a) $f(x) = \sqrt{x^2+1}$

(b) $g(x) = \frac{x^4-16x^2}{x+\sqrt{x}} + (x-2)\sqrt[3]{x+1}$

Solution

(a) $x^2+1 \geq 0$

x^2 is always $+$

$\therefore D_f: \mathbb{R}$

(b) $x+\sqrt{x} \neq 0 \cap x > 0$

$D_g: (0,\infty)$

تذكر

خصائص الجذور

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

$$\sqrt{xy} = \sqrt{x}\cdot\sqrt{y}$$

$$\sqrt{x+y} \neq \sqrt{x}+\sqrt{y}$$

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

خصائص الضرب والقسمة

$$a(b+c) = ab+ac$$

$$\frac{a+b}{c} = \frac{a}{c}+\frac{b}{c}, \quad c \neq 0$$

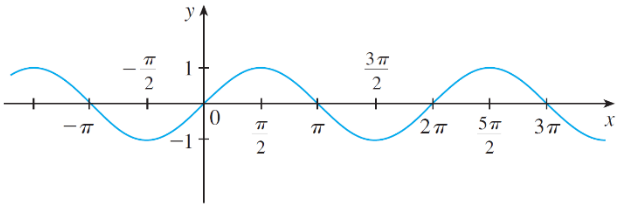
$$\frac{a}{c+b} \neq \frac{a}{c}+\frac{a}{b}$$

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm cb}{bd}$$

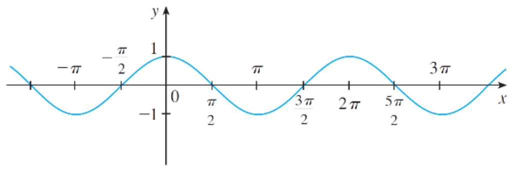
$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b}\cdot\frac{d}{c} = \frac{ad}{bc}$$

- Trigonometric functions الدوال المثلثية (6/8):

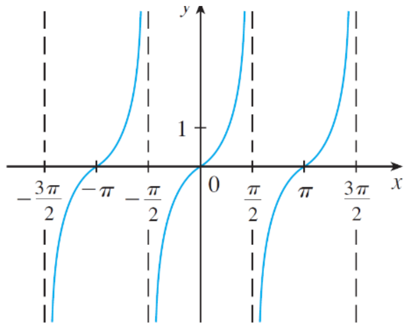
	Domain	Range
$\mathbf{sin}(\boldsymbol{x})$	$\mathbb{R} = (-\infty, \infty)$	$[-1,1]$
$\mathbf{cos}(\boldsymbol{x})$	$\mathbb{R} = (-\infty, \infty)$	$[-1,1]$
$\mathbf{tan}(\boldsymbol{x})$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\mathbb{R} = (-\infty, \infty)$



y = sin x



y = cos x



y = tan x

تذكر

$\sin\theta = \frac{y}{r} = \frac{opp}{hyp}$ المقابل الوتر

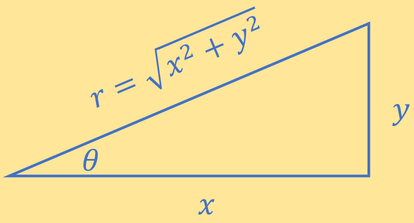
$\cos\theta = \frac{x}{r} = \frac{adj}{hyp}$ المجاور

$\tan\theta = \frac{y}{x} = \frac{opp}{adj}$

$\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$



$\sin(x + 2\pi) = \sin(x), \quad \forall x$

$\cos(x + 2\pi) = \cos(x), \quad \forall x$

$\tan(x + \pi) = \tan(x), \quad \forall x$

تذكر

Identities:

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

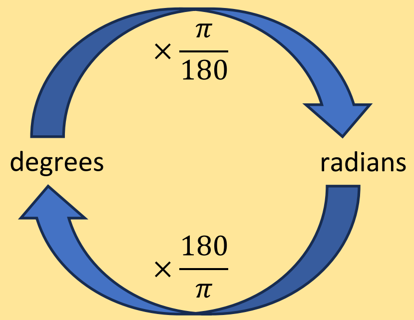
$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

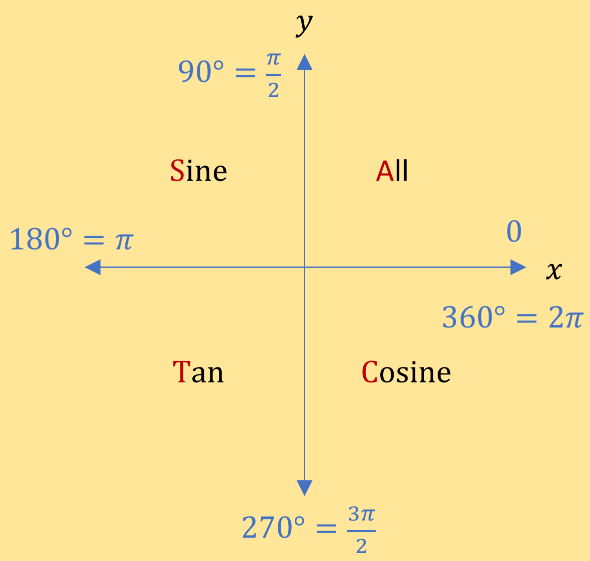
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - \sin^2\theta\end{aligned}$$

θ		$\sin\theta$	$\cos\theta$	$\tan\theta$
0		0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	DNE
$180°$	π	0	-1	0
$270°$	$\frac{3\pi}{2}$	-1	0	DNE
$360°$	2π	0	1	0

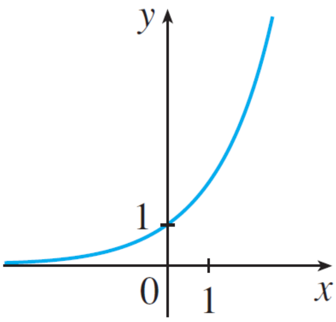




- Exponential functions الدوال الأسية (7/8):

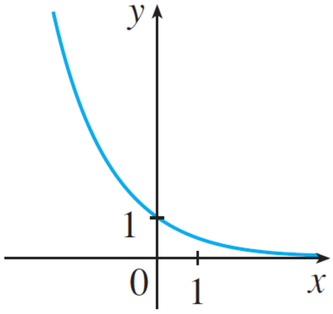
$$f(x) = y = a^x$$

$a > 1$



Domain: $\mathbb{R}$
Range: $(0, \infty)$

$0 < a < 1$



Domain: $\mathbb{R}$
Range: $(0, \infty)$

- Logarithmic functions الدوال اللوغاريتمية (8/8):

$f(x) = \log_a x = \frac{\ln x}{\ln a}, a \neq 1$

$D_f: (0, \infty)$

$R_f: \mathbb{R} = (-\infty, \infty)$

Example 3

Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \frac{1+x}{1-\sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

Solution

(a) Exponential

(b) Power

(c) Algebraic

(d) Polynomial, degree $= 4$

Problems

- Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

(a) $f(x) = \log_2 x$

Logarithmic

(b) $g(x) = \sqrt[4]{x}$

root

(c) $h(x) = \frac{2x^3}{1-x^2}$

Rational

(d) $u(t) = 1 - 1.1t + 2.54t^2$

Polynomial

(e) $v(t) = 5^t$

Exponential

(f) $w(\theta) = \sin\theta\cos^2\theta$

Trigonometric

- **Find the domain of the function.**

$$g(x) = \frac{1}{1 - \tan x}$$

$1 - \tan x = 0$

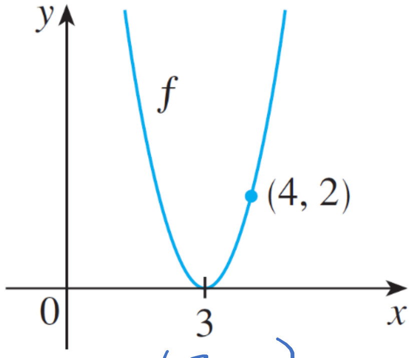
$\tan x = 1$

$x = \frac{\pi}{4}$

$D_g = \mathbb{R} / \{\pi/4\}$

- **Find expressions for the quadratic functions whose graphs are shown.**

(a) $f(x) = a(x-h)^2 + k$



$(3,0)$
(h,k)

$f(x) = a(x-3)^2 + 0$

$f(x) = a(x^2 - 6x + 9)$

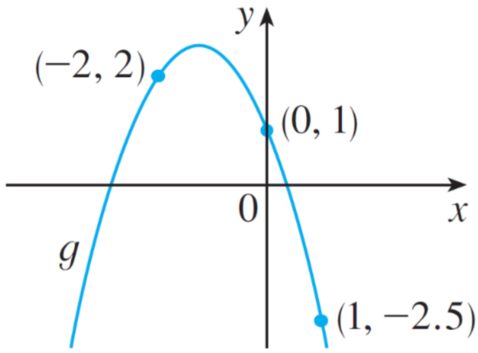
$2 = a(4^2 - 6(4) + 9)$

$2 = a(16 - 24 + 9)$

$a = 2$

$\therefore f(x) = 2(x^2 - 6x + 9 = 2x^2 - 12x + 18$

(b) $f(x) = ax^2 + bx + c$



$1 = a(0)^2 + b(0) + c$

$\boxed{c = 1}$

$-2.5 = a(1)^2 + b(1) + 1$

$-2.5 = a + b + 1$

$b = -2.5 - 1 - a$

$= -3.5 - a \longrightarrow (1)$

$2 = a(-2)^2 + b(-2) + 1$

$2 = 4a - 2b + 1$

$2 = 4a - 2(-3.5 - a) + 1$

$2 = 4a + 7 + 2a + 1$

$6a = 2 - 7 - 1$

$6a = -6 \Rightarrow \boxed{a = -1}$

$\boxed{b = -3.5 - (-1) = -2.5}$

$f(x) = -x^2 - 2.5x + 1$