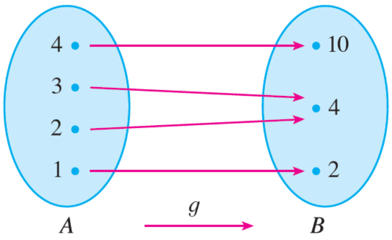
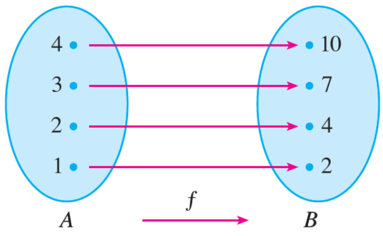
Section 1.5 – Inverse Functions and Logarithms

- One-to-one function:





A function f is called one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Example 1

Is the function one-to-one?

(a) $f(x) = x^2$ (b) $f(x) = x^3$

Solution

(a) $f(a) = f(b)$

$a^2 = b^2$

$\sqrt{a^2} = \sqrt{b^2}$

$a = \pm b$

Not one-to-one

(b) $f(a) = f(b)$

$a^3 = b^3$

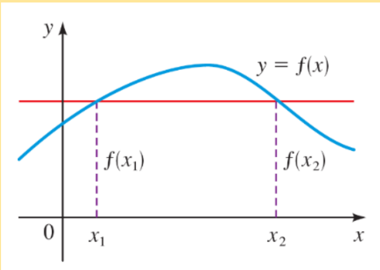
$\sqrt[3]{a^3} = \sqrt[3]{b^3}$

$a = b$

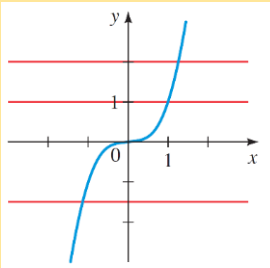
one-to-one

لاحظ

نقدر نعرف إذا الدالة one-to-one من الرسم برسم خط أفقي Horizontal Line Test



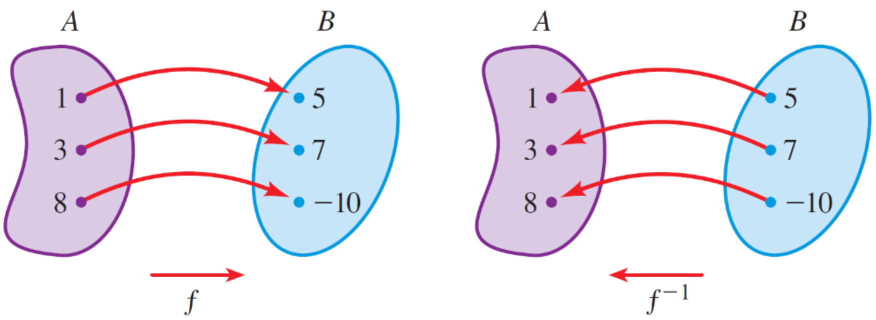
Not one-to one



One-to-one

- أي دالة one-to-one لها inverse function دالة عكسية

$f(x)=y$



$f^{-1}(y)=x$

ويكون

domain of f^{-1} = range of f

range of f^{-1} = domain of f

انتبه

-1 في f^{-1} ليست أس

$f^{-1}(x)$ لا تعني $\frac{1}{f(x)}$

- لأي دالة لها دالة عكسية inverse function

$$f(x)=y \quad \Leftrightarrow \quad f^{-1}(y)=x$$

Example 2

If $f(1)=5$, $f(3)=7$, and $f(8)=-10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

Solution

$f^{-1}(7)=3$

$f^{-1}(5)=1$

$f^{-1}(-10)=8$

- To **find the inverse of a function**

1- اكتب y مكان $f(x)$

2- نحل لإيجاد x كمعادلة في y

3- نضع x مكان y و $f^{-1}(x)$ مكان x

Example 3

Find the inverse function of $f(x) = x^3 + 2$

Solution

$y = x^3 + 2$ الخطوة 1

$x^3 = y - 2$

$x = \sqrt[3]{y - 2}$ الخطوة 2

$f^{-1}(x) = \sqrt[3]{x - 2}$ الخطوة 3

لاحظ

لأن المعتاد هو كتابة y بدلالة x $\Leftarrow y = f(x)$

ولأننا الآن مهتمين بدراسة الدالة العكسية inverse function

فسنقوم بعكس تعريف x و y حتى تظل y معرفة بدلالة x

$x = f(y) \xrightarrow{\text{بالتبديل}} f^{-1}(x) = y$

مع ملاحظة أن ذلك لا يغير أي قاعدة أو قانون، وإنما هو الالتزام بطريقة كتابة مألوفة

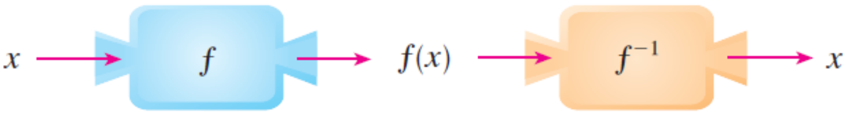
- **Cancellation equations**:

لأي دالة $f(x)$ لها $f^{-1}(x)$ inverse function

$$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$$

بشرط x تنتمي إلى Domain f^{-1} $\quad x \in D_{f^{-1}}$

$x \in D_f$ بشرط x تنتمي إلى Domain f



مثال:

$f(x) = 3x - 2$ and $f^{-1}(x) = \frac{x+2}{3}$

$$f\left(f^{-1}(x)\right) = \cancel{3}\left(\frac{x+2}{\cancel{3}}\right) - 2 = x + \cancel{2} - \cancel{2} = x$$

Example 4

If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f\left(f^{-1}(2)\right)$

Solution

$f(x) = 3$

$x^5 + x^3 + x = 3$

$1^5 + 1^3 + 1 = 3$

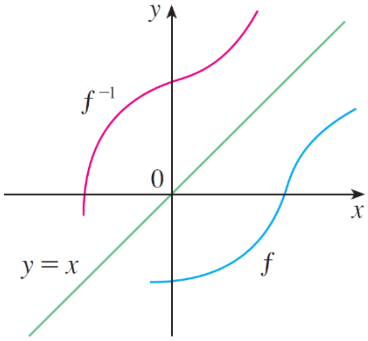
$\therefore f(1) = 3$

$\therefore f^{-1}(3) = 1$

$f\left(f^{-1}(2)\right) = 2$

by cancellation equations

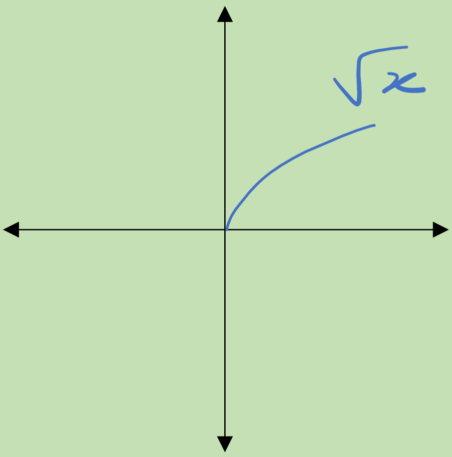
- نستطيع رسم الدالة العكسية inverse function بعكس رسم الدالة الأصلية حول الخط $y = x$

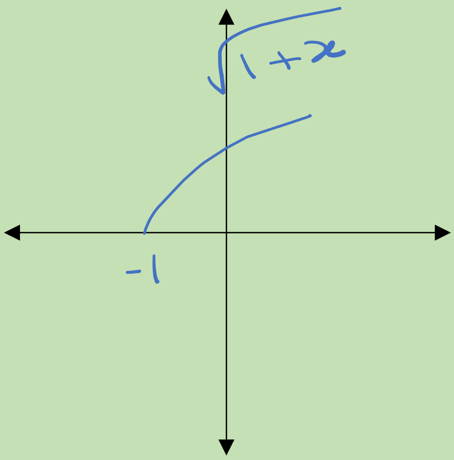


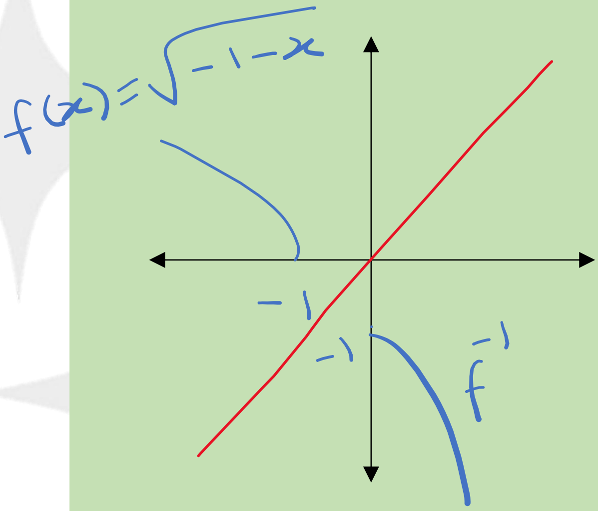
Example 5

Sketch the graphs of $f(x) = \sqrt{-1 - x}$ and its inverse function using the same coordinate axes

Solution







- الدالة اللوغاريتمية logarithmic function هي عكس inverse الدالة الأسية exponential function

Exponent الأس

$a^{y} = x$

Base الأساس

Exponent الأس

$\log_a x = y$

Base الأساس

Exponential function

نكتب الأساس و الأس على اليسار، والناتج على اليمين

Logarithmic function

نكتب الأساس و الناتج على اليسار، و الأس على اليمين

بشرط: الأساس $a > 0$ و $a \neq 1$

لاحظ

تطبيق cancellation equations على اللوغاريتم

$$\log_a(a^x) = x, \quad \forall x \in \mathbb{R}$$
$$a^{\log_a a} = x, \quad \forall x > 0$$

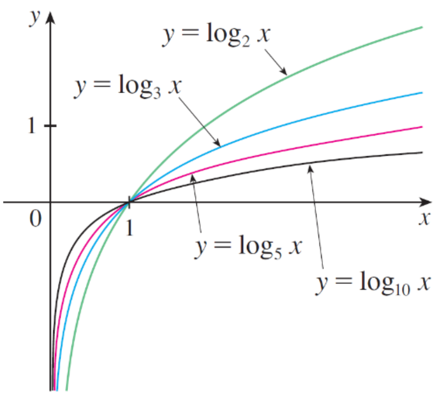
- For $y = f(x) = \log_a x$

domain: $(0, \infty) = \mathbb{R}^+$

x لازم تكون موجبة

range: $(-\infty, \infty) = \mathbb{R}$

y عادي تكون أي قيمة



- **Properties of logarithms**

1. $\log_a 1 = 0$

2. $\log_a a = 1$

3. $\log_a a^x = x$

4. $a^{\log_a x} = x$

- **Laws of logarithms**

1. $\log_a (xy) = \log_a x + \log_a y$

2. $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

3. $\log_a x^r = r \log_a x$

Example 6

Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

Solution

$$\log_2 80 - \log_2 5 = \log_2 \left(\frac{80}{5}\right)$$
$$= \log_2 16$$
$$= 4 \qquad \Rightarrow \; 2^4 = 16$$

لأن

- Natural Logarithms:

حالة خاصة من اللوغاريتمات يكون الأساس فيها هو العدد e

$\log_e x = \ln x$

نضع كلمة ln بدل كلمة log لكن لا يتغير أي شيء في القوانين والقواعد

$\ln x = y \quad \Longleftrightarrow \quad e^y = x$

- Properties of natural logarithms

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $e^{\ln x} = x$

- Laws of natural logarithms

1. $\ln(xy) = \ln x + \ln y$
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
3. $\ln x^r = r \ln x$

Example 7

Find the value of x for the following:

(a) $\ln x = 5$ (b) $e^{5-3x} = 10$ (c) $\ln x = \ln 30 + \ln 10$

Solution

(a) $\ln x = 5$

$e^5 = x$ من تعريف اللوغاريتم

(b) $e^{5-3x} = 10$

$\ln e^{5-3x} = \ln 10$ اخاصية رقم 3

$5 - 3x = \ln 10$

$5 - \ln 10 = 3x$

$x = \frac{5 - \ln 10}{3}$

(c) $\ln x = \ln 30 + \ln 10$ القانون الأول

$\ln x = \ln(30 \cdot 10)$

$\ln x = \ln(300)$

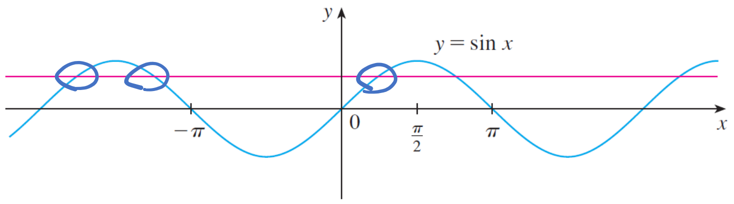
$x = 300$

لاحظ

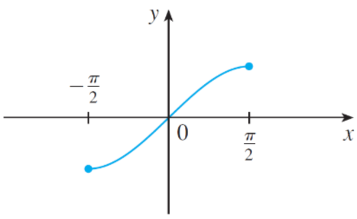
نستطيع تحويل أي لوغاريتم إلى natural logarithm، باستخدام change of base formula

$$\log_a x = \frac{\ln x}{\ln a}$$

- Inverse Trigonometric Functions:



الدالة $\sin x$ ليست one-to-one وبالتالي inverse function لها تكون فقط في الفترة $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



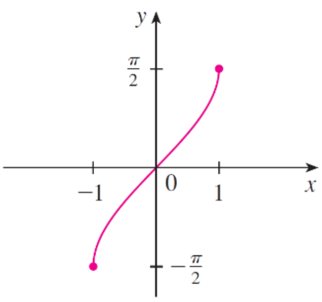
$D_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$R_f = [-1, 1]$

$\sin^{-1} x = \arcsin x = y \quad \Leftrightarrow \quad \sin y = x \qquad$ where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$D_{f^{-1}} = [-1, 1] = R_f$

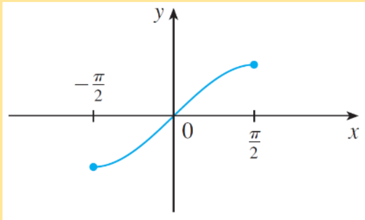
$R_{f^{-1}} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = D_f$



لاحظ

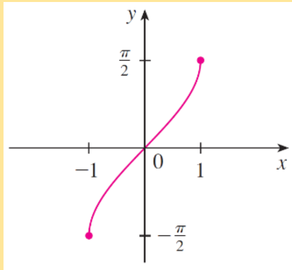
$\sin^{-1}(\sin x) = x \qquad \text{for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

domain
sin



$\sin(\sin^{-1}x) = x \qquad \text{for } -1 \le x \le 1$

domain
$\sin^{-1}$



Example 8

Evaluate

(a) $\sin^{-1}\left(\frac{1}{2}\right)$ (b) $\tan\left(\arcsin \frac{1}{3}\right)$

Solution

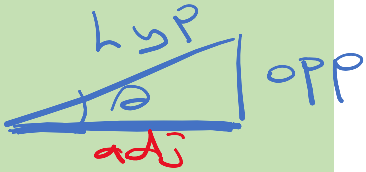
(a) $\sin^{-1}\left(\frac{1}{2}\right) = x \Rightarrow \sin x = \frac{1}{2}$

$x = \frac{\pi}{6} = 30°$

تكافئ

(b) $\arcsin \frac{1}{3} \equiv \sin^{-1}\frac{1}{3} = \theta$

$\sin \theta = \frac{1}{3}$



$\therefore \tan\left(\sin^{-1}\frac{1}{3}\right)$

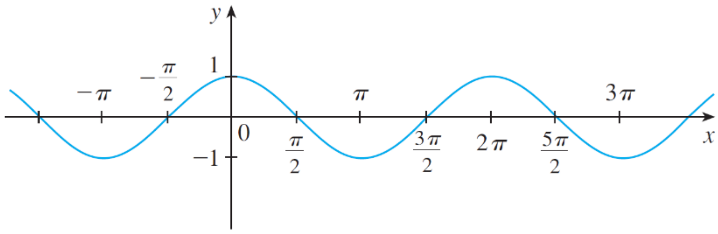
$= \tan(\theta) = \frac{opp}{adj} = \frac{1}{\sqrt{8}}$

$\sin\theta = \frac{opp}{hyp} = \frac{1}{3}$

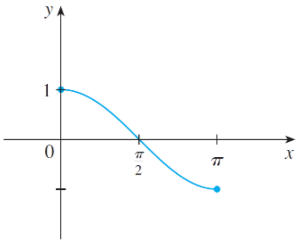
$adj = \sqrt{3^2 - 1^2}$

$= \sqrt{8}$

- The inverse cosine function



الدالة $\cos x$ ليست one-to-one وبالتالي inverse function لها تكون فقط في الفترة $[0,\pi]$



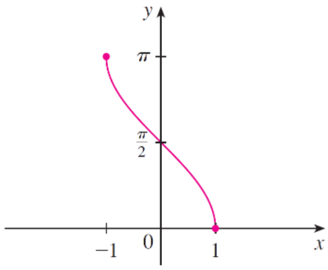
$D_f = [0, \pi]$

$R_f = [-1, 1]$

$\cos^{-1} x = y \iff \cos y = x \quad$ where $0 \le y \le \pi$

$D_{f^{-1}} = [-1, 1] = R_f$

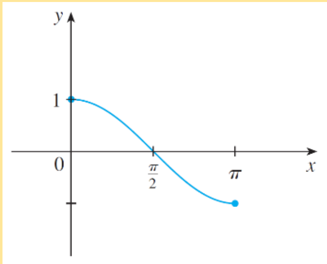
$R_{f^{-1}} = [0, \pi] = D_f$



لاحظ

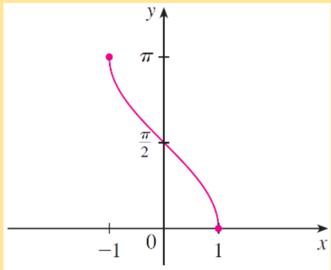
$\cos^{-1}(\cos x) = x \quad$ for $\quad 0 \le x \le \pi$

domain cos

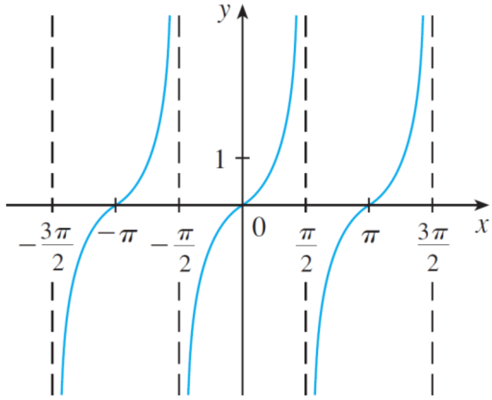


$\cos(\cos^{-1} x) = x \quad$ for $\quad -1 \le x \le 1$

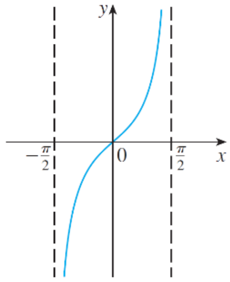
domain $\cos^{-1}$



- The inverse tangent function



الدالة $\tan x$ ليست one-to-one وبالتالي inverse function لها تكون فقط في الفترة $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$



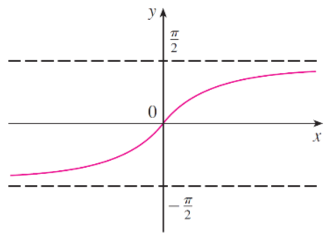
$D_f = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$R_f = \mathbb{R}$

$\tan^{-1} x = y \iff \tan y = x$ where $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$D_{f^{-1}} = \mathbb{R} = R_f$

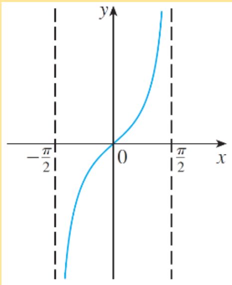
$R_{f^{-1}} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = D_f$



لاحظ

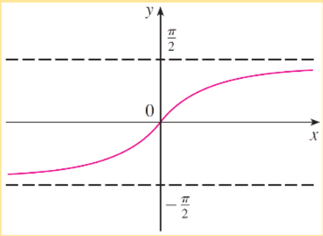
$\tan^{-1}(\tan x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

domain tan



$\tan(\tan^{-1} x) = x$ for $\forall x \in \mathbb{R}$

domain $\tan^{-1}$



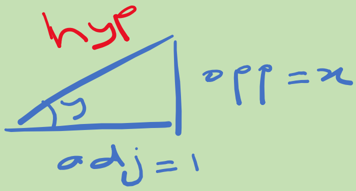
Example 9

Simplify the expression $\cos(\tan^{-1} x)$.

Solution

$\tan^{-1} x = y$

$\tan y = x$



$\text{hyp} = \sqrt{x^2+1}$

$\therefore \cos(\tan^{-1} x)$

$= \cos(y) = \dfrac{1}{\sqrt{x^2+1}}$

- Inverse of other trigonometric functions

$\csc^{-1} x = y \iff \csc y = x$

$|x| \geq 1$ $\qquad$ $y \in (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$

$\sec^{-1} x = y \iff \sec y = x$

$|x| \geq 1$ $\qquad$ $y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

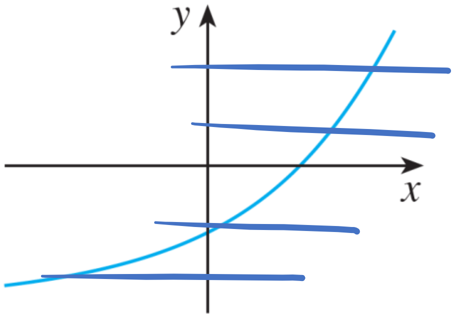
$\cot^{-1} x = y \iff \cot y = x$

$x \in \mathbb{R}$ $\qquad$ $y \in (0, \pi)$

Problems

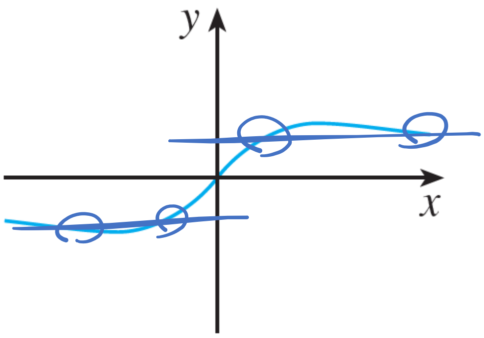
- Determine whether the function is one-to-one

(a)



One-to-one

(b)



Not one-to-one

(c) $f(x) = x^4 - 16$

$$f(a) = f(b)$$

$$a^4 - 16 = b^4 - 16$$

$$a^4 = b^4 \quad \cancel{-16} \ \cancel{+16}$$

$$\sqrt[4]{a^4} = \pm\sqrt[4]{b^4}$$

$$a = \pm b$$

Not one-to-one

- Find a formula for the inverse of the function, and find its domain and range

(a) $f(x) = \frac{4x-1}{2x+3}$

$$y = \frac{4x-1}{2x+3}$$

$$y(2x+3) = 4x-1$$

$$2xy + 3y = 4x - 1$$

$$2xy - 4x = -3y - 1$$

$$x(2y-4) = -3y-1$$

$$x = \frac{-3y-1}{2y-4}$$

$$f^{-1}(x) = \frac{-3x-1}{2x-4}$$

Domain :

$$2x+3=0$$

$$2x=-3$$

$$x=-\frac{3}{2}$$

$$D_f = \mathbb{R} / \{-3/2\}$$

$$R_f = D_{f^{-1}}$$

$$2x-4=0$$

$$2x=4$$

$$x=\frac{4}{2}=2$$

$$\boxed{R_f = \mathbb{R} / \{2\}}$$

(b) $y = \frac{1-e^{-x}}{1+e^{-x}}$

$$y(1+e^{-x}) = 1-e^{-x}$$

$$y + ye^{-x} = 1-e^{-x}$$

$$ye^{-x} + e^{-x} = 1-y$$

$$e^{-x}(y+1) = 1-y$$

$$e^{-x} = \frac{1-y}{y+1} \Rightarrow \frac{1}{e^{x}} = \frac{1-y}{y+1}$$

$$e^{x} = \frac{y+1}{1-y}$$

$$x = \ln\left(\frac{y+1}{1-y}\right) \Rightarrow f^{-1}(x) = \ln\left(\frac{x+1}{1-x}\right)$$

D_f :

$$1+e^{-x} = 0$$

$$e^{-x} = -1$$

$$\frac{1}{e^{x}} = \frac{-1}{1} \Rightarrow e^{x} = -1 \qquad \text{No solution}$$

$$D_f = \mathbb{R}$$

$$R_f = D_{f^{-1}}$$

$$\frac{x+1}{1-x} > 0$$

zeros: $\{-1, 1\}$

		-1		1	
$x+1$	$-$		$+$		$+$
$1-x$	$+$		$+$		$-$
	$-$		$+$		$-$

$$D_{f^{-1}} = (-1, 1)$$

(c) $f(x) = \sqrt{1 - x^2}, 0 \le x \le 1.$

$y = \sqrt{1 - x^2}$

$y^2 = 1 - x^2$

$x^2 = 1 - y^2$

$x = \sqrt{1 - y^2}$ or $x = -\sqrt{1 - y^2}$ X
rejected
because $0 \le x \le 1$

$f^{-1}(x) = \sqrt{1 - x^2}$

$D_f = [0, 1]$

$R_f = [0, 1]$

- Find the exact value of each expression.

(a) $\log_5 \frac{1}{125}$

$$= \log_5 \frac{1}{5^3}$$

$$= \log_5 5^{-3} = -3$$

(b) $\ln(1/e^2)$

$$= \ln e^{-2} = -2$$

(c) $e^{-\ln 2}$

$$= \frac{1}{e^{\ln 2}} = \frac{1}{2}$$

بالخاصية الرابعة

$$e^{\ln x} = x$$

حل آخر

$$e^{-\ln 2} = e^{-1 \cdot \ln 2}$$

$$= e^{\ln 2^{-1}}$$

$$= e^{\ln \frac{1}{2}}$$

$$= \frac{1}{2}$$

(d) $e^{\ln(\ln e^3)}$

$$e^{\ln(\ln e^3)} = \ln e^3 = 3$$

$$e^{\ln x} = x \qquad \ln e^x = x$$

- Express the given quantity as a single logarithm.

$$\frac{1}{3}\ln(x+2)^3 + \frac{1}{2}[\ln x - \ln(x^2+3x+2)^2]$$

$$= \ln[(x+2)^3]^{1/3} + \frac{1}{2}\ln x - \frac{1}{2}\ln(x^2+3x+2)^2$$

$$= \ln(x+2) + \ln x^{1/2} - \ln[(x^2+3x+2)^2]^{1/2}$$

$$= \ln(x+2)\cdot x^{1/2} - \ln(x^2+3x+2)$$

$$= \ln \frac{(x+2)\sqrt{x}}{x^2+3x+2}$$

- **Solve each equation for x.**

(a) $\ln(x^2 - 1) = 3$

$e^3 = x^2 - 1$

$x^2 = e^3 + 1$

$x = \pm\sqrt{e^3 + 1}$

(b) $e^{2x} - 3e^x + 2 = 0$

$(e^x)^2 - 3e^x + 2 = 0$

$(e^x - 2)(e^x - 1) = 0$

$e^x = 2$ or $e^x = 1$

$\ln e^x = \ln 2$ $\qquad$ $\ln e^x = \ln 1$

$x = \ln 2$ $\qquad$ $x = 0$

- (a) Find the domain of $f(x) = \ln(e^x - 3)$.

$$e^x - 3 > 0$$

$$e^x > 3$$

$$\ln e^x > \ln 3$$

$$x > \ln 3$$

$$D_f = \{x \mid x > \ln 3\}$$

(b) Find f^{-1} and its domain.

$$y = \ln(e^x - 3)$$

$$e^y = e^x - 3$$

$$e^x = e^y + 3$$

$$x = \ln(e^y + 3)$$

$$f^{-1}(x) = \ln(e^x + 3)$$

$$e^x + 3 > 0$$

$$e^x > -3$$

$$D_{f^{-1}} = \mathbb{R}$$

- Find the exact value of each expression.

(a) $\tan^{-1}\sqrt{3}$

$$\tan^{-1}\sqrt{3} = x$$

$$\tan x = \sqrt{3}$$

$$x = \frac{\pi}{3} = 60^\circ$$

(b) $\arctan(-1)$

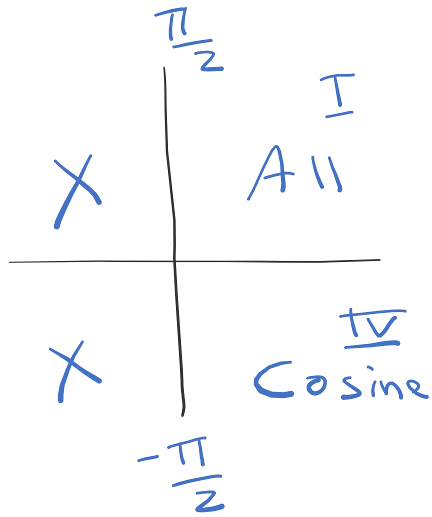
$$\tan^{-1}(-1) = x$$

$$\tan x = -1$$

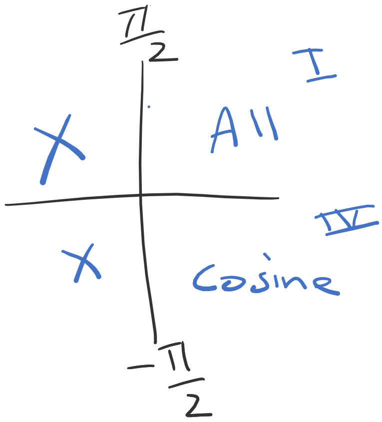
x in quadrant $\underline{\text{IV}}$

$\because \tan 45 = 1$

$\therefore x = -45^\circ = -\frac{\pi}{4}$



(c) $\sin^{-1}(-1/\sqrt{2})$



$$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = x$$

$$\sin x = -\frac{1}{\sqrt{2}}$$

x in quadrant IV

$$\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad x = -\frac{\pi}{4} = -45^\circ$$

(d) $\cos^{-1}(\sqrt{3}/2)$

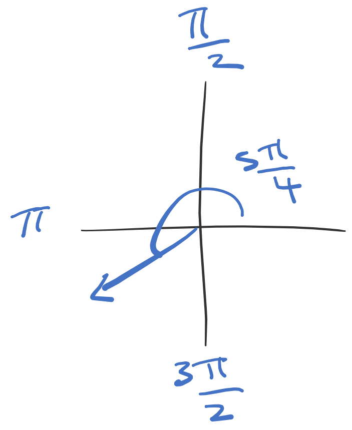
$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = x$$

$$\cos x = \sqrt{\frac{3}{2}}$$

$$x = \frac{\pi}{6} = 30^\circ$$

(e) arcsin(sin(5π/4))

$\frac{5\pi}{4}$ is in quadrant III



$$\bar{\theta} = \theta - \pi$$

$$= \frac{5\pi}{4} - \pi = \frac{\pi}{4}$$

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\therefore \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

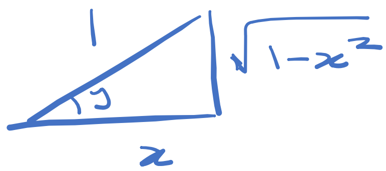
$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4} = -45^\circ$$

- Simplify the expression

$$\sin(2 \arccos x)$$

$$\cos^{-1} x = y$$

$$\cos y = x$$



$$\sin y = \sqrt{1-x^2}$$

$$\sin(2\cos^{-1} x) = \sin(2y)$$

$$= 2\sin y \cos y$$

$$= 2\sqrt{1-x^2} \cdot x$$