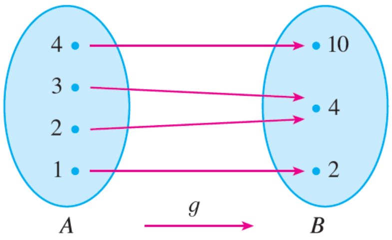
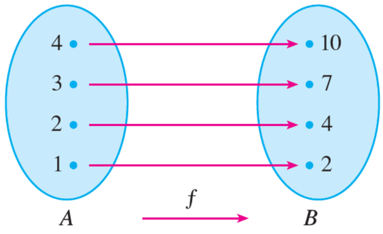
Section 1.5 – Inverse Functions and Logarithms

- One-to-one function:





A function f is called one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Example 1

Is the function one-to-one?

(a) $f(x) = x^2$ (b) $f(x) = x^3$

Solution

(a) $f(a) = f(b)$

$a^2 = b^2$

$\sqrt{a^2} = \sqrt{b^2}$

$a = \pm b$

Not one-to-one

(b) $f(a) = f(b)$

$a^3 = b^3$

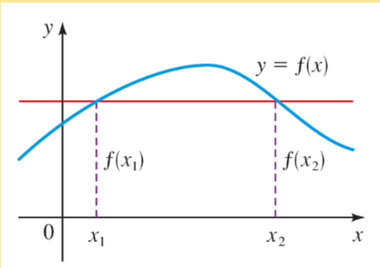
$\sqrt[3]{a^3} = \sqrt[3]{b^3}$

$a = b$

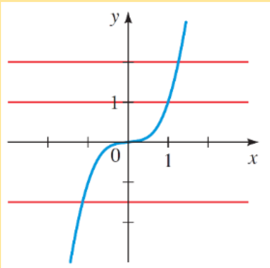
one-to-one

لاحظ

نقدر نعرف إذا الدالة one-to-one من الرسم برسم خط أفقي Horizontal Line Test



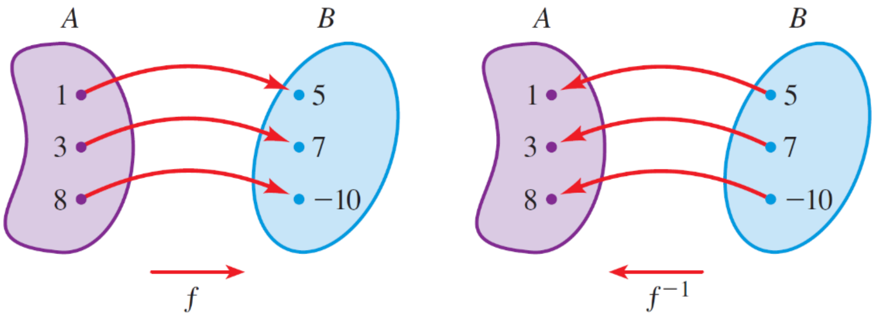
Not one-to one



One-to-one

- أي دالة one-to-one لها inverse function دالة عكسية

$f(x)=y$



$f^{-1}(y)=x$

ويكون

domain of f^{-1} = range of f

range of f^{-1} = domain of f

انتبه

-1 في f^{-1} ليست أس

$f^{-1}(x)$ لا تعني $\frac{1}{f(x)}$

- لأي دالة لها دالة عكسية inverse function

$$f(x) = y \quad \Leftrightarrow \quad f^{-1}(y) = x$$

Example 2

If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

Solution

$f^{-1}(7) = 3$

$f^{-1}(5) = 1$

$f^{-1}(-10) = 8$

- To **find the inverse of a function**

1- اكتب y مكان $f(x)$

2- نحل لإيجاد x كمعادلة في y

3- نضع x مكان y و $f^{-1}(x)$ مكان x

Example 3

Find the inverse function of $f(x) = x^3 + 2$

Solution

$y = x^3 + 2$ ① الخطوة

$x^3 = y - 2$

$x = \sqrt[3]{y - 2}$ ② الخطوة

$f^{-1}(x) = \sqrt[3]{x - 2}$ ③ الخطوة

لاحظ

لأن المعتاد هو كتابة y بدلالة x ⇐ $y = f(x)$

ولأننا الآن مهتمين بدراسة الدالة العكسية inverse function

فسنقوم بعكس تعريف x و y حتى تظل y معرفة بدلالة x

$x = f(y) \xrightarrow{\text{تصبح}} f^{-1}(x) = y$

مع ملاحظة أن ذلك لا يغير أي قاعدة أو قانون، وإنما هو الالتزام بطريقة كتابة مألوفة

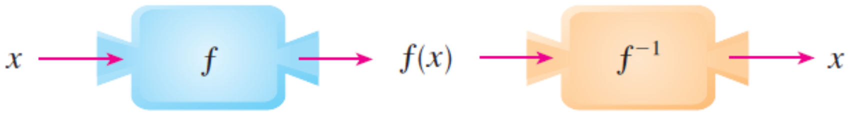
- **Cancellation equations**:

لأي دالة $f(x)$ لها $f^{-1}(x)$ inverse function

$$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$$

$x \in D_{f^{-1}}$ بشرط x تنتمي إلى Domain f^{-1}

$x \in D_f$ بشرط x تنتمي إلى Domain f



مثال:

$f(x) = 3x - 2$ and $f^{-1}(x) = \frac{x+2}{3}$

$$f\left(f^{-1}(x)\right) = 3\left(\frac{x+2}{3}\right) - 2 = x + 2 - 2 = x$$

Example 4

If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f\left(f^{-1}(2)\right)$

Solution

$f(x) = 3$

$x^5 + x^3 + x = 3$

$1^5 + 1^3 + 1 = 3$

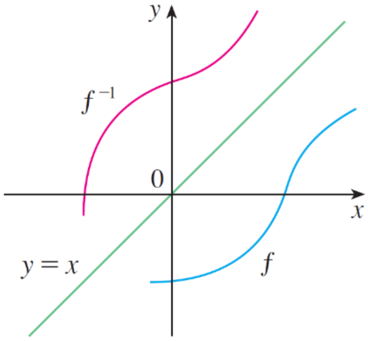
$\therefore f(1) = 3$

$\therefore f^{-1}(3) = 1$

$f\left(f^{-1}(2)\right) = 2$

by cancellation equations

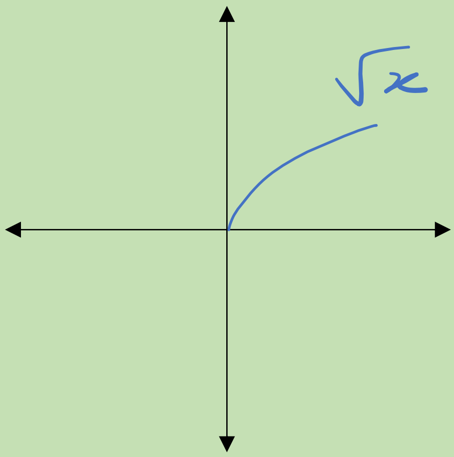
- نستطيع رسم الدالة العكسية inverse function بعكس رسم الدالة الأصلية حول الخط $y = x$

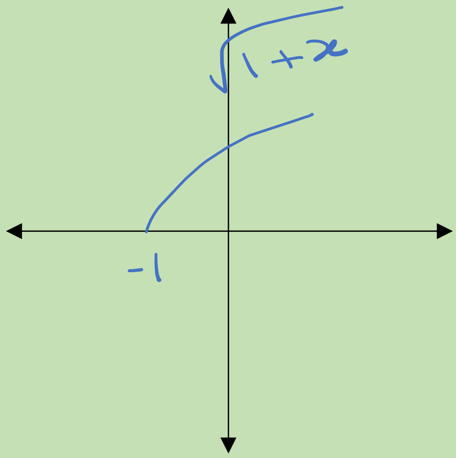


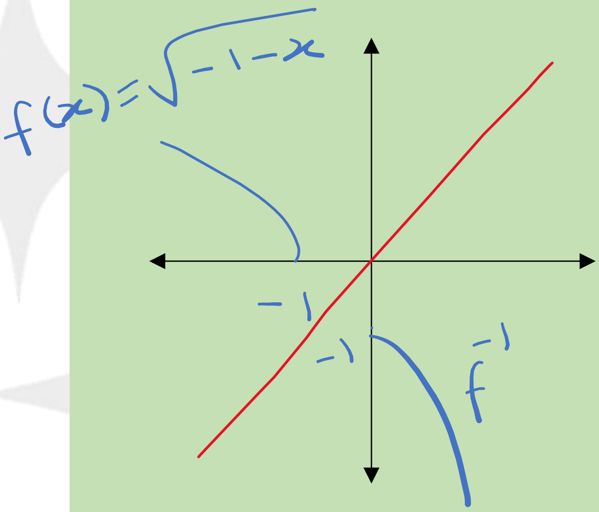
Example 5

Sketch the graphs of $f(x) = \sqrt{-1 - x}$ and its inverse function using the same coordinate axes

Solution







- الدالة اللوغاريتمية logarithmic function هي عكس inverse الدالة الأسية exponential function

Exponent الأس

$\log_a x = y$

Base الأساس

Exponent الأس

$a^y = x$

Base الأساس

Logarithmic function

نكتب الأساس و الناتج على اليسار، و الأس على اليمين

Exponential function

نكتب الأساس و الأس على اليسار، والناتج على اليمين

<u>بشرط: الأساس $a > 0$ و $a \neq 1$</u>

لاحظ

تطبيق cancellation equations على اللوغاريتم

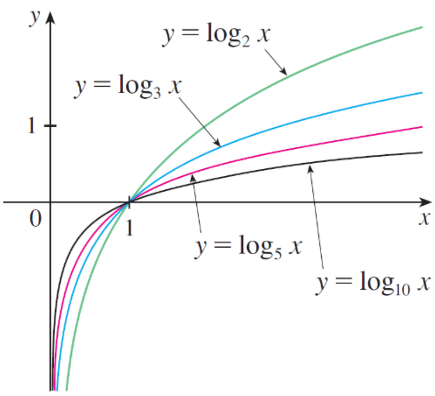
$$\log_a(a^x) = x, \quad \forall x \in \mathbb{R}$$
$$a^{\log_a a} = x, \quad \forall x > 0$$

- For $y = f(x) = \log_a x$
domain: $(0, \infty) = \mathbb{R}^+$

x لازم تكون موجبة

range: $(-\infty, \infty) = \mathbb{R}$

y عادي تكون أي قيمة



- **Properties of logarithms**

1. $\log_a 1 = 0$

2. $\log_a a = 1$

3. $\log_a a^x = x$

4. $a^{\log_a x} = x$

- **Laws of logarithms**

1. $\log_a(xy) = \log_a x + \log_a y$

2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

3. $\log_a x^r = r\log_a x$

Example 6

Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

Solution

$$\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right)$$

$$= \log_2 16$$

$$= 4 \qquad \Rightarrow \quad 2^4 = 16$$

لأن

- Natural Logarithms:

حالة خاصة من اللوغاريتمات يكون الأساس فيها هو العدد e

$\log_e x = \ln x$

نضع كلمة ln بدل كلمة log لكن لا يتغير أي شيء في القوانين والقواعد

$\ln x = y \quad \Leftrightarrow \quad e^y = x$

- Properties of natural logarithms

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^{x} = x$
4. $e^{\ln x} = x$

- Laws of natural logarithms

1. $\ln(xy) = \ln x + \ln y$
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
3. $\ln x^{r} = r \ln x$

Example 7

Find the value of x for the following:

(a) $\ln x = 5$ (b) $e^{5-3x} = 10$ (c) $\ln x = \ln 30 + \ln 10$

Solution

(a) $\ln x = 5$

$e^{5} = x$ من تعريف اللوغاريتم

(b) $e^{5-3x} = 10$

$\ln e^{5-3x} = \ln 10$ اخذ صيغة رقم 3

$5 - 3x = \ln 10$

$5 - \ln 10 = 3x$

$x = \frac{5 - \ln 10}{3}$

(c) $\ln x = \ln 30 + \ln 10$ القانون الأول

$\ln x = \ln(30 \cdot 10)$

$\ln x = \ln(300)$

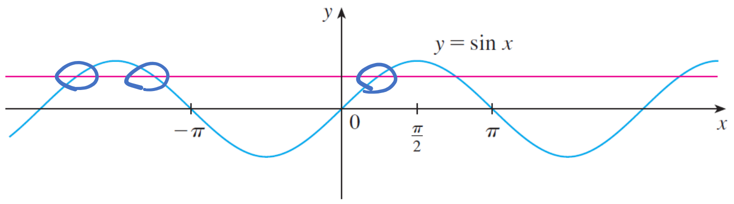
$x = 300$

لاحظ

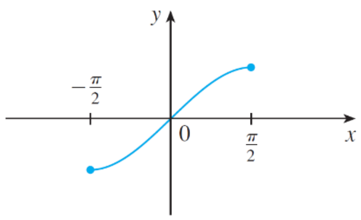
نستطيع تحويل أي لوغاريتم إلى natural logarithm، باستخدام change of base formula

$$\log_a x = \frac{\ln x}{\ln a}$$

- Inverse Trigonometric Functions:



الدالة $\sin x$ ليست one-to-one وبالتالي inverse function لها تكون فقط في الفترة $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



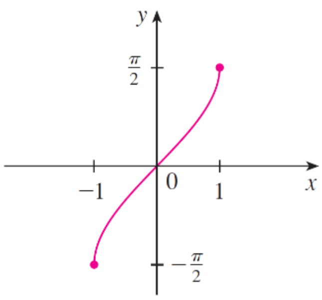
$D_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$R_f = [-1, 1]$

$\sin^{-1} x = \arcsin x = y \iff \sin y = x \quad$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$D_{f^{-1}} = [-1, 1] = R_f$

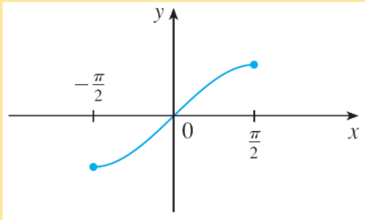
$R_{f^{-1}} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = D_f$



لاحظ

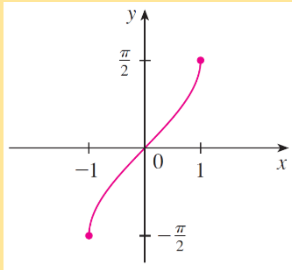
$\sin^{-1}(\sin x) = x \qquad \text{for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

domain sin



$\sin(\sin^{-1} x) = x \qquad \text{for } -1 \le x \le 1$

domain $\sin^{-1}$



Example 8

Evaluate

(a) $\sin^{-1}\left(\frac{1}{2}\right)$ (b) $\tan\left(\arcsin \frac{1}{3}\right)$

Solution

(a) $\sin^{-1}\left(\frac{1}{2}\right) = x \Rightarrow \sin x = \frac{1}{2}$

$x = \frac{\pi}{6} = 30°$

(b) $\arcsin \frac{1}{3} \equiv \sin^{-1} \frac{1}{3} = \theta$ يكافئ

$\sin \theta = \frac{1}{3}$

$\therefore \tan\left(\sin^{-1} \frac{1}{3}\right)$

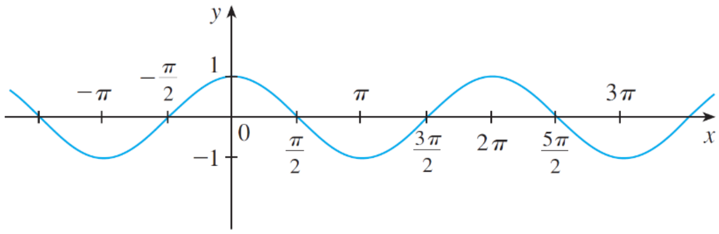
$= \tan(\theta) = \frac{opp}{adj} = \frac{1}{\sqrt{8}}$

$\sin \theta = \frac{opp}{hyp} = \frac{1}{3}$

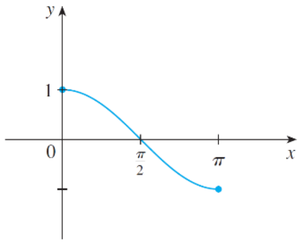
$adj = \sqrt{3^2 - 1^2}$

$= \sqrt{8}$

- The inverse cosine function



الدالة $\cos x$ ليست one-to-one وبالتالي inverse function لها تكون فقط في الفترة $[0, \pi]$



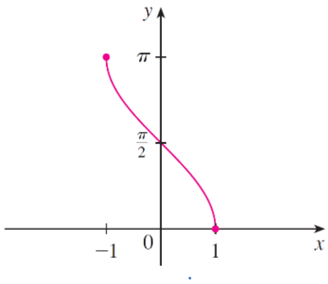
$D_f = [0, \pi]$

$R_f = [-1, 1]$

$\cos^{-1} x = y \iff \cos y = x$ where $0 \le y \le \pi$

$D_{f^{-1}} = [-1, 1] = R_f$

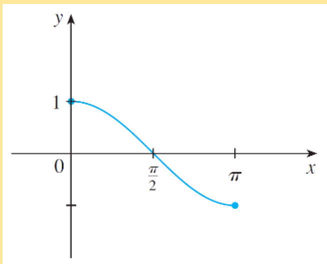
$R_{f^{-1}} = [0, \pi] = D_f$



لاحظ

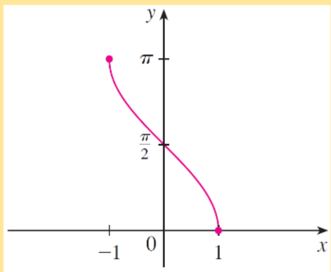
$\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$

domain cos

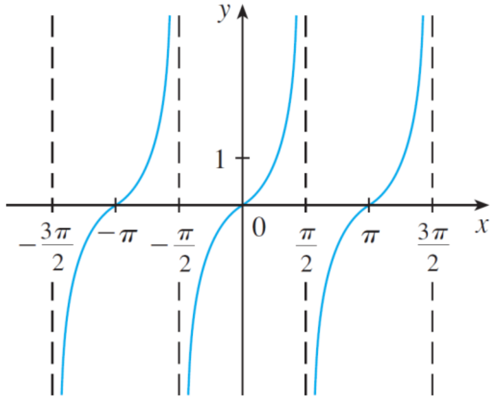


$\cos(\cos^{-1} x) = x$ for $-1 \le x \le 1$

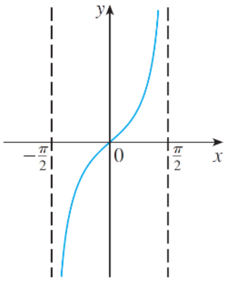
domain $\cos^{-1}$



- The inverse tangent function



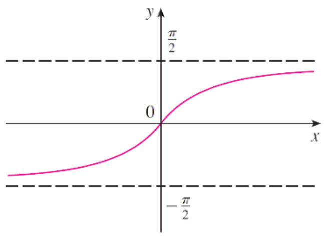
الدالة $\tan x$ ليست one-to-one وبالتالي inverse function لها تكون فقط في الفترة $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$D_f = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$R_f = \mathbb{R}$

$\tan^{-1} x = y \iff \tan y = x \quad$ where $-\frac{\pi}{2} < y < \frac{\pi}{2}$



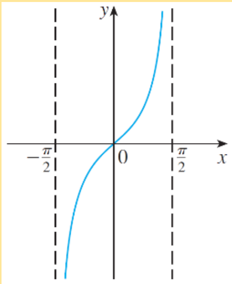
$D_{f^{-1}} = \mathbb{R} = R_f$

$R_{f^{-1}} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = D_f$

لاحظ

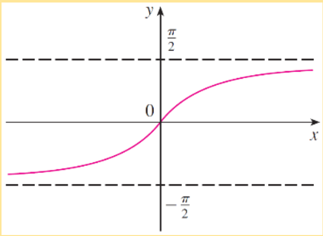
$\tan^{-1}(\tan x) = x \quad$ for $\quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

domain tan



$\tan(\tan^{-1} x) = x \quad$ for $\quad \forall x \in \mathbb{R}$

domain $\tan^{-1}$



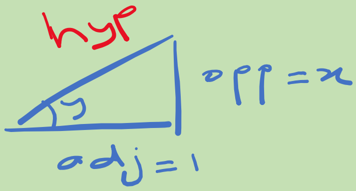
Example 9

Simplify the expression $\cos(\tan^{-1} x)$.

Solution

$\tan^{-1} x = y$

$\tan y = x$



hyp $= \sqrt{x^2+1}$

$\therefore \cos(\tan^{-1} x)$

$= \cos(y) = \dfrac{1}{\sqrt{x^2+1}}$

- Inverse of other trigonometric functions

$\csc^{-1} x = y \iff \csc y = x$

$|x| \geq 1$ $\qquad$ $y \in (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$

$\sec^{-1} x = y \iff \sec y = x$

$|x| \geq 1$ $\qquad$ $y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

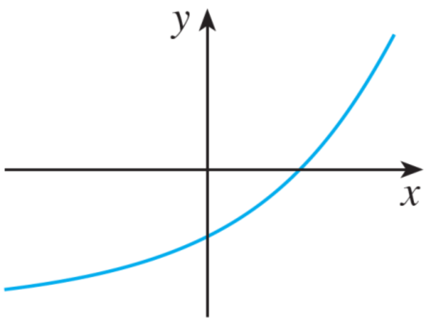
$\cot^{-1} x = y \iff \cot y = x$

$x \in \mathbb{R}$ $\qquad$ $y \in (0, \pi)$

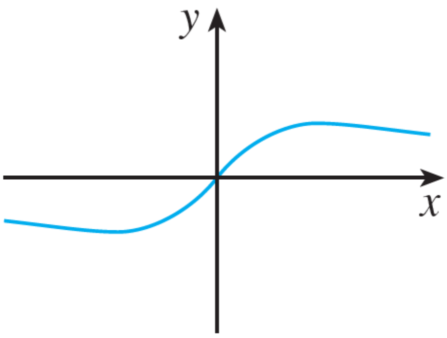
Problems

- Determine whether the function is one-to-one

(a)



(b)



(c) $f(x) = x^4 - 16$

- Find a formula for the inverse of the function

(a) $f(x) = \frac{4x-1}{2x+3}$

(b) $y = \frac{1-e^{-x}}{1+e^{-x}}$

***f* sxd − s1 2 *x*2 , 0 < *x* < 1.**

- Find the exact value of each expression.

(a) $\log_5 \frac{1}{125}$

(b) $\ln(1/e^2)$

(c) $e^{-\ln 2}$

(d) $e^{\ln(\ln e^3)}$

- Express the given quantity as a single logarithm.

$$\frac{1}{3}\ln(x+2)^3 + \frac{1}{2}[\ln x - \ln(x^2+3x+2)^2]$$

- Solve each equation for x.

(a) $\ln(x^2 - 1) = 3$

(b) $e^{2x} - 3e^x + 2 = 0$

- (a) Find the domain of $f(x) = \ln(e^x - 3)$.

(b) Find f^{-1} and its domain.

- Find the exact value of each expression.

(a) $\tan^{-1}\sqrt{3}$

(b) $\arctan(-1)$

(c) $\sin^{-1}(-1/\sqrt{2})$

(d) $\cos^{-1}(\sqrt{3}/2)$

(e) $\arcsin(\sin(5\pi/4))$

- Simplify the expression

$$\sin(2 \arccos x)$$