Section 2.2 – The Limit of a Function

- The limit of a function نهاية الدالة:

$$\lim_{x \to a} f(x) = L$$

$x \to a \quad \Rightarrow \quad f(x) \to L$

x approaches "a" as $f(x)$ approaches L

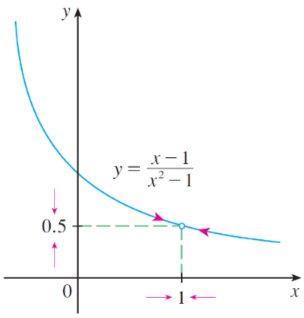
The values of $f(x)$ gets arbitrarily close to L by restricting x to be sufficiently close to a

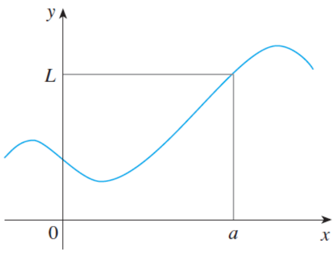
كلما اقتربت x من قيمة a ، اقتربت $f(x)$ من قيمة L

مثال

$f(x) \to 0.5$ as $x \to 1$

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$$

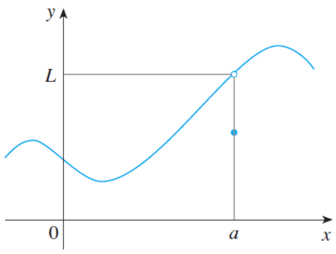




(a)

$f(a) = L$

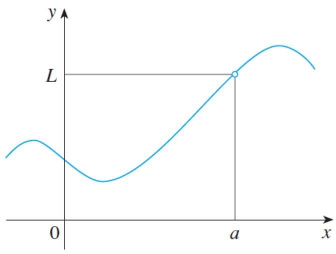
$\lim_{x \to a} f(x) = L$



(b)

$f(a) \neq L$

$\lim_{x \to a} f(x) = L$



(c)

$f(a)$ undefined

$\lim_{x \to a} f(x) = L$

* قيمة الدالة ليس لها علاقة بقيمة limit

Example 1

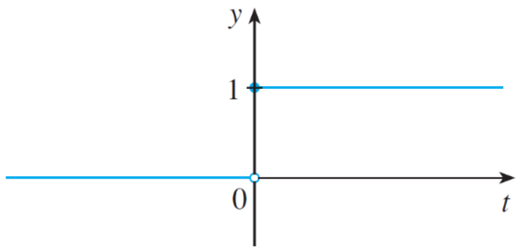
Find $\lim_{x\to 2} x^2 - 2x$

Solution

$$\lim_{x\to 2} x^2 - 2x = 2^2 - 2(2) = 4 - 4 = 0$$

- One-Sided Limits

$$H(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0 \end{cases}$$



$$\lim_{t\to 0^{+}} H(t) = 1$$

من اليمين

$$\lim_{t\to 0^{-}} H(t) = 0$$

من اليسار

لاحظ

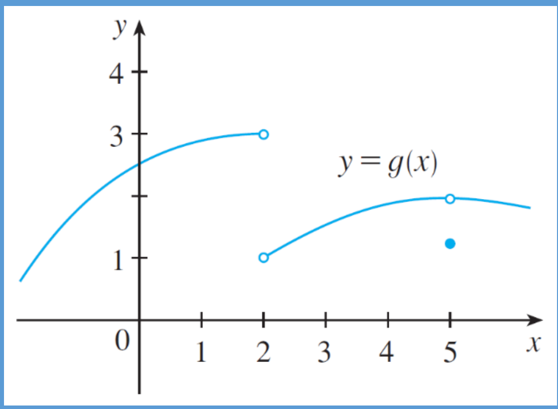
$$\lim_{x\to a} f(x) = L$$

فقط إذا كان اليمين و اليسار نفس القيمة

$$\lim_{x\to a^{-}} f(x) = \lim_{x\to a^{+}} f(x) = L$$

Example 2

The graph of a function g is shown. Use it to state the values (if they exist) of the following:



(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(a) $\lim_{x \to 5^-} g(x)$ (b) $\lim_{x \to 5^+} g(x)$ (c) $\lim_{x \to 5} g(x)$

Solution

(a) $\lim_{x \to 2^-} g(x) = 3$

(b) $\lim_{x \to 2^+} g(x) = 1$

(c) $\lim_{x \to 2} g(x) = \text{DNE}$

لأن

$\boxed{\lim_{x \to 2^-} \neq \lim_{x \to 2^+}}$

DNE = Does Not Exist

ليس لها وجود

(d) $\lim_{x \to 5^-} g(x) = 2$

(e) $\lim_{x \to 5^+} g(x) = 2$

(f) $\lim_{x \to 5} g(x) = 2$

Example 3

- For the function f, state the value of each quantity if it exists. If it doesn't exist, explain why.

$$f(x) = \begin{cases} 1+x & \text{if} \quad x < -1 \\ x^2 & \text{if} \quad -1 \le x < 1 \\ 2-x & \text{if} \quad x \ge 1 \end{cases}$$

(a) $\lim_{x \to -1^-} f(x)$ (b) $\lim_{x \to -1^+} f(x)$ (c) $\lim_{x \to -1} f(x)$

(d) $\lim_{x \to 1^-} f(x)$ (e) $\lim_{x \to 1^+} f(x)$ (f) $\lim_{x \to 1} f(x)$

Solution

(a) $\lim_{x \to -1^-} 1 + x = 1 + (-1) = 0$

(b) $\lim_{x \to -1^+} x^2 = (-1)^2 = 1$

(c) $\lim_{x \to -1} f(x) = \text{DNE}$

because $\lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$

(d) $\lim_{x \to 1^-} x^2 = 1^2 = 1$

(e) $\lim_{x \to 1^+} 2 - x = 2 - 1 = 1$

(f) $\lim_{x \to 1} f(x) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$= 1$

Example 4

Find the limit of the function.

$$\lim_{x\to4}\frac{|x-4|}{x-4}$$

Solution

$$|x-4| = \begin{cases} x-4 & x \geq 4 \\ -(x-4) & x < 4 \end{cases}$$

$$\lim_{x\to4^-} \frac{-\cancel{(x-4)}}{\cancel{x-4}} = -1$$

$$\lim_{x\to4^+} \frac{x-4}{x-4} = 1$$

$$\therefore \lim_{x\to4} \frac{|x-4|}{x-4} = DNE$$

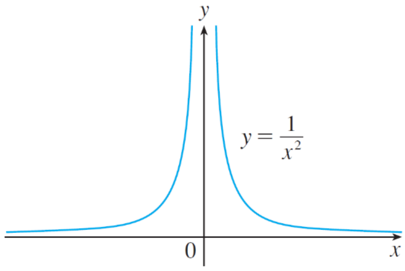
لاحظ

إذا كان للدالة أكثر من تعريف ومطلوب lim عند نقطة تغير التعريف لاااااازم نحسب من الناحيتين حتى لو مش مطلوب.

- Infinite Limits

$$\lim_{x \to 0} \frac{1}{x^2} = \frac{1}{0} = \infty$$

أي ثابت تقسيم صفر $\left.\begin{matrix} \infty \\ \text{أو} \\ -\infty \end{matrix}\right.$

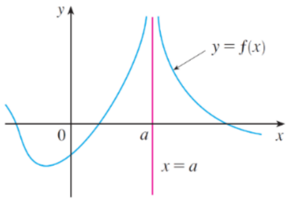


$x = a$ is a vertical asymptote لكل الحالات التالية

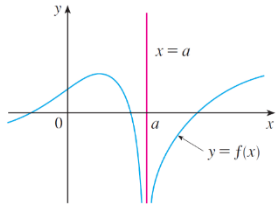
$\lim_{x \to a} f(x) = \infty$ $\quad$ $\lim_{x \to a} f(x) = -\infty$

$\lim_{x \to a^-} f(x) = \infty$ $\quad$ $\lim_{x \to a^-} f(x) = -\infty$

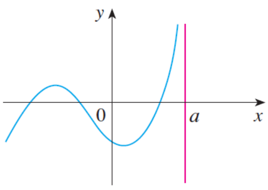
$\lim_{x \to a^+} f(x) = \infty$ $\quad$ $\lim_{x \to a^+} f(x) = -\infty$



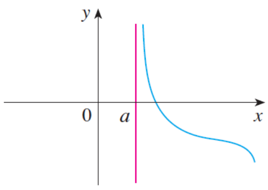
$\lim_{x \to a} f(x) = \infty$



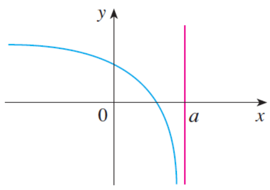
$\lim_{x \to a} f(x) = -\infty$



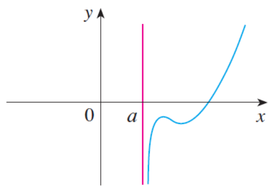
(a) $\lim_{x \to a^-} f(x) = \infty$



(b) $\lim_{x \to a^+} f(x) = \infty$



(c) $\lim_{x \to a^-} f(x) = -\infty$



(d) $\lim_{x \to a^+} f(x) = -\infty$

Example 5

Find $\lim_{x \to 3^+} \frac{2x}{x-3}$ and $\lim_{x \to 3^-} \frac{2x}{x-3}$

Solution

$$\lim_{x \to 3^+} \frac{6}{0^+} = \infty$$

$$\lim_{x \to 3^-} \frac{6}{0^-} = -\infty$$

Example 6

Find the vertical asymptotes of the function $\ln x$.

Solution

$$\lim_{x \to 0} \ln x = -\infty$$

$$\boxed{e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0}$$

$x = 0$ is vertical asymptote for $\ln x$

Example 7

Find the limit of the function.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Solution

$$\lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}$$

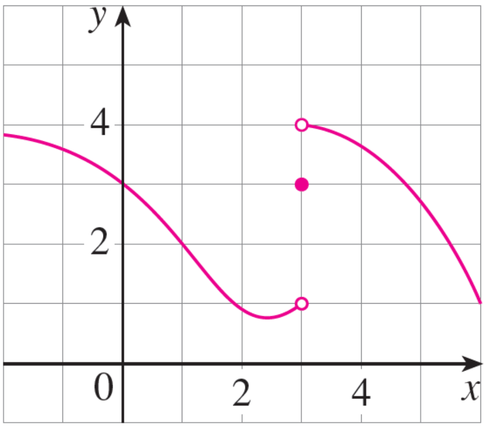
$$= \lim_{x \to 1} x + 1 = 1 + 1 = 2$$

لاحظ لو بدأنا بالتعويض $\frac{1^2-1}{1-1} = \frac{1-1}{1-1} = \frac{0}{0}$

دايما نبدأ بالاختصار إن أمكن

Problems

- For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.



(a) $\lim_{x\to1} f(x) = 2$

(b) $\lim_{x\to3^-} f(x) = 1$

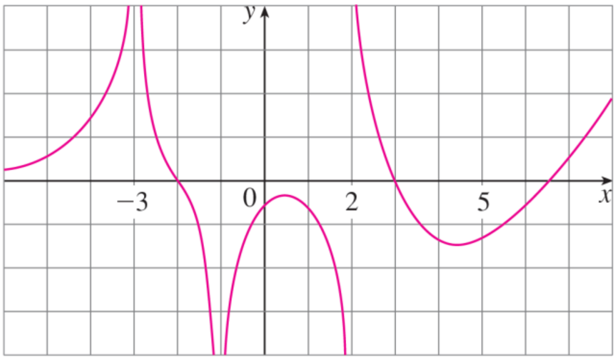
(c) $\lim_{x\to3^+} f(x) = 4$

(d) $\lim_{x\to3} f(x) = \text{DNE}$

because $\lim_{x\to3^-} f(x) \neq \lim_{x\to3^+} f(x)$

(e) $f(3) = 3$

- **For the function A whose graph is given, state the following.**



(a) $\lim_{x \to -3} A(x)$ $= \infty$

(b) $\lim_{x \to 2^-} A(x)$ $= -\infty$

(c) $\lim_{x \to 2^+} A(x)$ $= \infty$

(d) $\lim_{x \to -1} A(x)$ $= -\infty$

(e) The equations of the vertical asymptotes

$x = -3$

$x = -1$

$x = 2$

- For the function f, state the value of each quantity if it exists. If it doesn't exist, explain why.

$$f(x) = \begin{cases} \dfrac{x+2}{x-1} & \text{if} \quad 0 < x \leq 4 \\ \sqrt{x} & \text{if} \quad x > 4 \end{cases}$$

(a) $\lim_{x \to 4^-} f(x) = \dfrac{4+2}{4-1}$

$= \dfrac{6}{3} = 2$

(b) $\lim_{x \to 4^+} f(x) = \sqrt{4} = 2$

(c) $\lim_{x \to 4} f(x) =$ DNE

(d) $\lim_{x \to 1} f(x) = \dfrac{1+2}{1-1}$

$= \dfrac{3}{0} = \infty$

- Find the infinite limit

(a) $\lim\limits_{x\to 5^-} \frac{x+1}{x-5}$

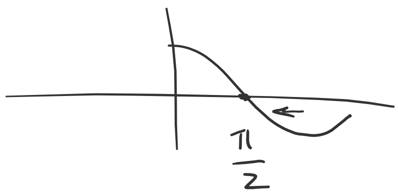
$$= \frac{6}{0^-} = -\infty$$

(b) $\lim\limits_{x\to 3^-} \frac{\sqrt{x}}{(x-3)^5}$

$$= \frac{\sqrt{3}}{0^-} = -\infty$$

ظلت القيمة سالبة
لأن الأس فردي
odd

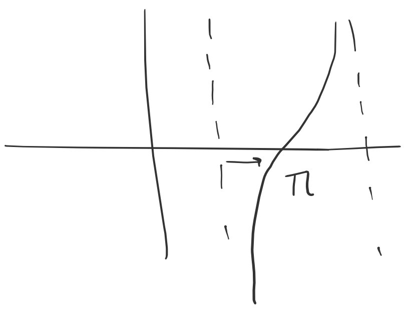
(c) $\lim\limits_{x\to(\pi/2)^+} \frac{1}{x}\sec x$



$$= \lim_{x\to(\pi/2)^+} \frac{1}{x} \frac{1}{\cos x}$$

$$= \frac{2}{\pi} \frac{1}{\cos(\pi/2)} = \frac{2}{0^-} = -\infty$$

(d) $\lim\limits_{x\to\pi^-} \cot x$



$$= \lim_{x\to\pi^-} \frac{1}{\tan x}$$

$$= \frac{1}{\tan\pi} = \frac{1}{0^-} = -\infty$$

- Find the vertical asymptotes of the function

$$y = \frac{x^2 + 1}{3x - 2x^2}$$

$$3x - 2x^2 = 0$$

$$x(3 - 2x) = 0$$

$x = 0$ or $3 - 2x = 0$

$2x = 3$

$x = \frac{3}{2}$

Vertical asymptotes

هي أصفار المقام

بشرط ⇐ الاقي أبسط صورة

يعني لا يكون اختصار شيء من البسط والمقام