

Section 2.2 – The Limit of a Function

- نهاية الدالة نهاية الدالة:

$$\lim_{x \rightarrow a} f(x) = L$$

$x \rightarrow a \Rightarrow f(x) \rightarrow L$
 x approaches "a" as $f(x)$ approaches L

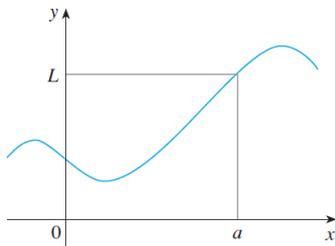
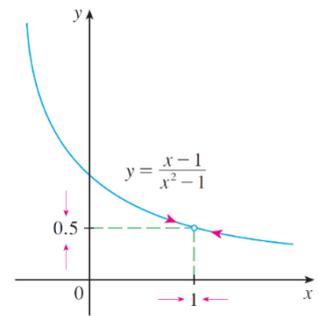
The values of $f(x)$ gets arbitrarily close to L by restricting x to be sufficiently close to a

كلما اقتربت x من قيمة a ، اقتربت $f(x)$ من قيمة L

مثال

$$f(x) \rightarrow 0.5 \text{ as } x \rightarrow 1$$

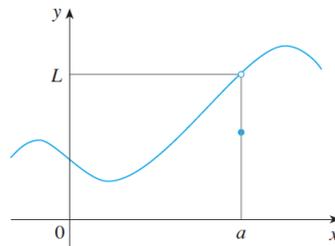
$$\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = 0.5$$



(a)

$$f(a) = L$$

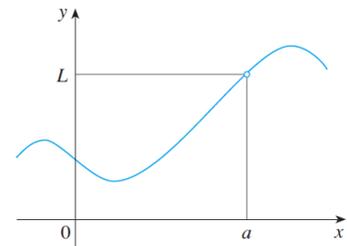
$$\lim_{x \rightarrow a} f(x) = L$$



(b)

$$f(a) \neq L$$

$$\lim_{x \rightarrow a} f(x) = L$$



(c)

$$f(a) \text{ undefined}$$

$$\lim_{x \rightarrow a} f(x) = L$$

* قيمة الدالة ليس لها علاقة بـ limit

Example 1

Find $\lim_{x \rightarrow 2} x^2 - 2x$

Solution

$$\lim_{x \rightarrow 2} x^2 - 2x = 2^2 - 2(2) = 4 - 4 = 0$$

- One-Sided Limits

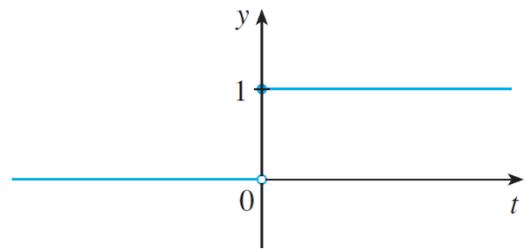
$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

$$\lim_{t \rightarrow 0^+} H(t) = 1$$

من اليمين

$$\lim_{t \rightarrow 0^-} H(t) = 0$$

من اليسار



لاحظ

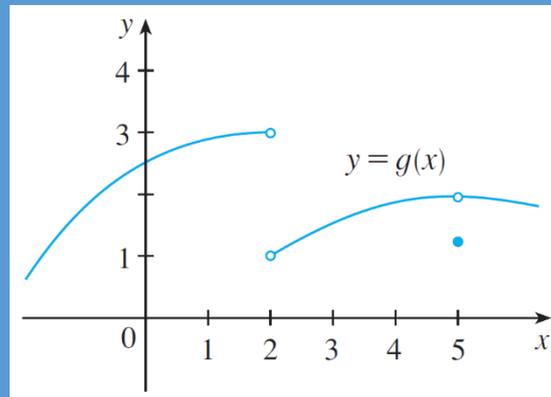
$$\lim_{x \rightarrow a} f(x) = L$$

فقط، اذ كان اليمين واليسار نفس القيمة

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

Example 2

The graph of a function g is shown. Use it to state the values (if they exist) of the following:



(a) $\lim_{x \rightarrow 2^-} g(x)$

(b) $\lim_{x \rightarrow 2^+} g(x)$

(c) $\lim_{x \rightarrow 2} g(x)$

(a) $\lim_{x \rightarrow 5^-} g(x)$

(b) $\lim_{x \rightarrow 5^+} g(x)$

(c) $\lim_{x \rightarrow 5} g(x)$

Solution

(a) $\lim_{x \rightarrow 2^-} g(x) = 3$

(b) $\lim_{x \rightarrow 2^+} g(x) = 1$

(c) $\lim_{x \rightarrow 2} g(x) = \text{DNE}$

(d) $\lim_{x \rightarrow 5^-} g(x) = 2$

(e) $\lim_{x \rightarrow 5^+} g(x) = 2$

(f) $\lim_{x \rightarrow 5} g(x) = 2$

$$\lim_{x \rightarrow 2^-} g(x) \neq \lim_{x \rightarrow 2^+} g(x) \quad \text{DNE}$$

↑

DNE = Does Not Exist

→ لا يوجد

Example 3

- For the function f , state the value of each quantity if it exists. If it doesn't exist, explain why.

$$f(x) = \begin{cases} 1+x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x < 1 \\ 2-x & \text{if } x \geq 1 \end{cases}$$

(a) $\lim_{x \rightarrow -1^-} f(x)$

(b) $\lim_{x \rightarrow -1^+} f(x)$

(c) $\lim_{x \rightarrow -1} f(x)$

(d) $\lim_{x \rightarrow 1^-} f(x)$

(e) $\lim_{x \rightarrow 1^+} f(x)$

(f) $\lim_{x \rightarrow 1} f(x)$

Solution

$$(a) \lim_{x \rightarrow -1^-} 1+x = 1+(-1) = 0$$

$$(b) \lim_{x \rightarrow -1^+} x^2 = (-1)^2 = 1$$

$$(c) \lim_{x \rightarrow -1} f(x) = \text{DNE}$$

$$\text{because } \lim_{x \rightarrow -1^-} f(x) \neq \lim_{x \rightarrow -1^+} f(x)$$

$$(d) \lim_{x \rightarrow 1^-} x^2 = 1^2 = 1$$

$$(e) \lim_{x \rightarrow 1^+} 2-x = 2-1 = 1$$

$$(f) \lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = 1$$

Example 4

Find the limit of the function.

$$\lim_{x \rightarrow 4} \frac{|x - 4|}{x - 4}$$

Solution

$$|x - 4| = \begin{cases} x - 4 & x \geq 4 \\ -(x - 4) & x < 4 \end{cases}$$

$$\lim_{x \rightarrow 4^-} \frac{-(x - 4)}{x - 4} = -1$$

$$\lim_{x \rightarrow 4^+} \frac{x - 4}{x - 4} = 1$$

$$\therefore \lim_{x \rightarrow 4} \frac{|x - 4|}{x - 4} = \text{DNE}$$

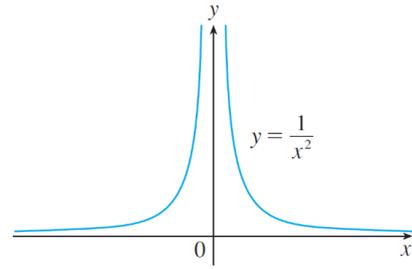
لاحظ

إذا كان للدالة أكثر من تعريف ومطلوب \lim عند نقطة تغير التعريف لالازم نحسب من الناحيتين حتى لو مش مطلوب.

- Infinite Limits

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \frac{1}{0} = \infty$$

أي ثابت تقسيم صفراً
 ∞ أو -∞



$x = a$ is a **vertical asymptote** لكل الحالات التالية

$$\lim_{x \rightarrow a} f(x) = \infty$$

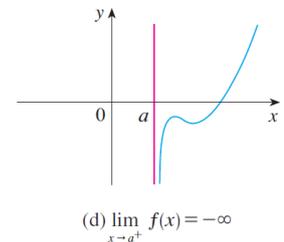
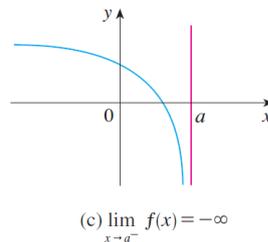
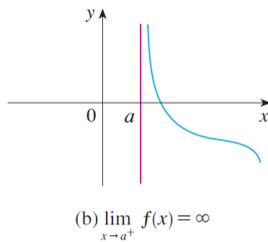
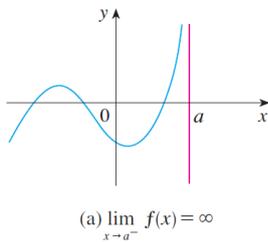
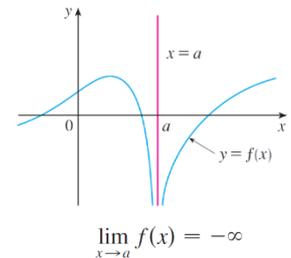
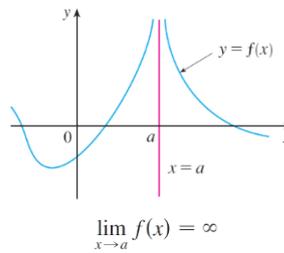
$$\lim_{x \rightarrow a} f(x) = -\infty$$

$$\lim_{x \rightarrow a^-} f(x) = \infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$



Example 5

Find $\lim_{x \rightarrow 3^+} \frac{2x}{x-3}$ and $\lim_{x \rightarrow 3^-} \frac{2x}{x-3}$

Solution

$$\lim_{x \rightarrow 3^+} \frac{6}{0^+} = \infty$$

$$\lim_{x \rightarrow 3^-} \frac{6}{0^-} = -\infty$$

Example 6

Find the vertical asymptotes of the function $\ln x$.

Solution

$$\lim_{x \rightarrow 0} \ln x = -\infty$$

$$e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

$x=0$ is vertical asymptote for $\ln x$

Example 7

Find the limit of the function.

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

Solution

$$\lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x \rightarrow 1} x + 1 = 1 + 1 = 2$$

$$\frac{0}{0} = \frac{1-1}{1-1} = \frac{1^2-1}{1-1}$$

لاحظ لو بد أن بالتعريف

دائماً تبدأ بالاضمحلال إن أمكن

Problems

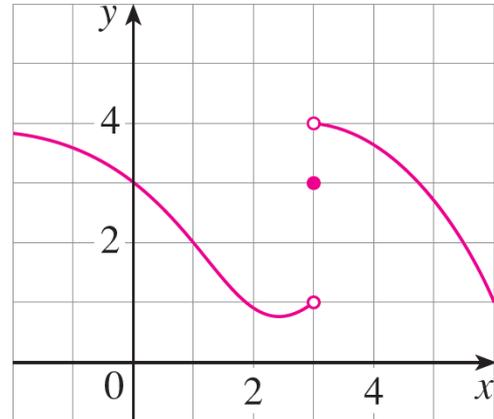
- For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \rightarrow 1} f(x)$

(b) $\lim_{x \rightarrow 3^-} f(x)$

(c) $\lim_{x \rightarrow 3^+} f(x)$

(d) $\lim_{x \rightarrow 3} f(x)$



(e) $f(3)$

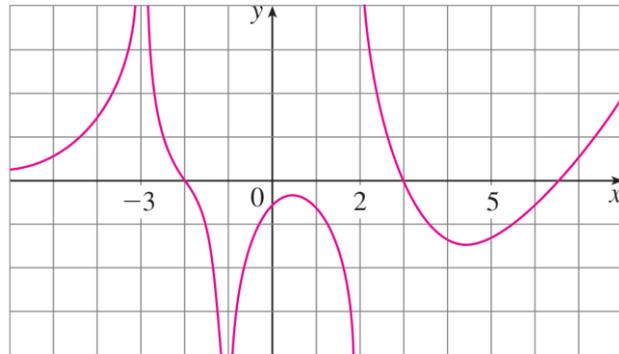
- For the function A whose graph is given, state the following.

(a) $\lim_{x \rightarrow -3} A(x)$

(b) $\lim_{x \rightarrow 2^-} A(x)$

(c) $\lim_{x \rightarrow 2^+} A(x)$

(d) $\lim_{x \rightarrow -1} A(x)$



(e) The equations of the vertical asymptotes

- For the function f , state the value of each quantity if it exists. If it doesn't exist, explain why.

$$f(x) = \begin{cases} \frac{x+2}{x-1} & \text{if } 0 < x \leq 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

(a) $\lim_{x \rightarrow 4^-} f(x)$

(b) $\lim_{x \rightarrow 4^+} f(x)$

(c) $\lim_{x \rightarrow 4} f(x)$

(d) $\lim_{x \rightarrow 1} f(x)$

- Find the infinite limit

(a) $\lim_{x \rightarrow 5^-} \frac{x+1}{x-5}$

(b) $\lim_{x \rightarrow 3^-} \frac{\sqrt{x}}{(x-3)^5}$

(c) $\lim_{x \rightarrow (\pi/2)^+} \frac{1}{x} \sec x$

(d) $\lim_{x \rightarrow \pi^-} \cot x$

- Find the vertical asymptotes of the function

$$y = \frac{x^2 + 1}{3x - 2x^2}$$