Section 2.3 – Calculating Limits Using the Limits Laws

- Limit Laws:

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \quad \text{(1)}$$ الجمع

$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \quad \text{(2)}$$ الطرح

$$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x) \quad \text{(3)}$$ ضرب ثابت

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x) \quad \text{(4)}$$ ضرب داليتين

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0 \quad \text{(5)}$$ القسمة

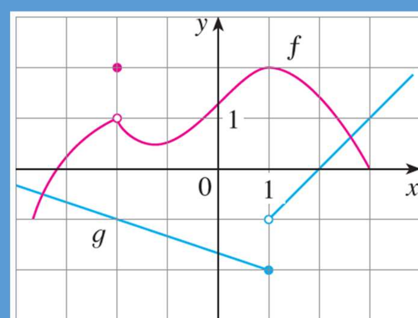
Example 1

Use the Limit Laws and the graphs of f and g in the figure to evaluate the following limits, if they exist.

(a) $\lim_{x\to -2}[f(x)+5g(x)]$

(b) $\lim_{x\to 1}[f(x)g(x)]$

(c) $\lim_{x\to 2}\frac{f(x)}{g(x)}$



Solution

(a) $\lim_{x\to -2}[f(x)+5g(x)]$

$= \lim_{x\to -2} f(x) + 5\lim_{x\to -2} g(x) = 1 + 5\cdot(-1) = -4$

(b) $\lim_{x\to 1}[f(x)g(x)] = \lim_{x\to 1} f(x) \lim_{x\to 1} g(x)$

$=$ DNE, because $\lim_{x\to 1} g(x)$ DNE

(c) $\lim_{x\to 2}\frac{f(x)}{g(x)} =$ DNE, because $\lim_{x\to 2} g(x) = 0$

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n \quad (6)$$

$$\lim_{x \to a} c = c \quad (7)$$

$$\lim_{x \to a} x = a \quad (8)$$

$$\lim_{x \to a} x^n = a^n \quad (9)$$

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \quad (10)$$

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \quad (11)$$

Example 2

Evaluate the following limits and justify each step.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$ (b) $\lim_{x \to 5} \frac{x^3+2x^2-1}{5-3x}$

Solution

(a) $\lim_{x \to 5} 2x^2 - \lim_{x \to 5} 3x + \lim_{x \to 5} 4$

$= 2\lim_{x \to 5} x^2 - 3\lim_{x \to 5} x + \lim_{x \to 5} 4$

$= 2 \cdot 5^2 - 3 \cdot 5 + 4 = 39$

(b) $\dfrac{\lim_{x \to 5} x^3 + 2x^2 - 1}{\lim_{x \to 5} 5 - 3x} = \dfrac{\lim_{x \to 5} x^3 + 2\lim_{x \to 5} x^2 - \lim_{x \to 5} 1}{\lim_{x \to 5} 5 - 3\lim_{x \to 5} x}$

$= \dfrac{5^3 + 2 \cdot 5^2 - 1}{5 - 3 \cdot 5} = \dfrac{174}{-10} = -17.4$

Example 3

Find $\lim_{x\to 1}\frac{x^2-1}{x-1}$

Solution

$$\lim_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}$$

التعويض المباشر لا يصلح $\frac{0}{0}$

$$= \lim_{x\to 1} x+1 = 1+1 = 2$$

لاحظ

أشهر طرق الاختصار

1- Expand ضرب

$$\lim_{h\to 0}\frac{(3+h)^2-9}{h} = \lim_{h\to 0}\frac{9+6h+h^2-9}{h} = \lim_{h\to 0}\frac{h(6+h)}{h} = \lim_{h\to 0} 6+h = 6$$

2- Factor تحليل

$$\lim_{x\to 0}\frac{x^2+x-6}{x-2} = \lim_{x\to 0}\frac{(x+3)(x-2)}{x-2} = \lim_{x\to 0} x+3 = 3$$

3- Conjugate المرافق

$$\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\cdot\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} = \lim_{t\to 0}\frac{(t^2+9)-9}{t^2(\sqrt{t^2+9}+3)}$$

$$= \lim_{t\to 0}\frac{t^2}{t^2\left(\sqrt{t^2+9}+3\right)} = \lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3} = \frac{1}{3+3} = \frac{1}{6}$$

مقام مشترك

4- Common denominator

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x(x+1)}\right) = \lim_{x\to 0}\frac{(x+1)-1}{x(x+1)} = \lim_{x\to 0}\frac{x}{x(x+1)} = \lim_{x\to 0}\frac{1}{x+1} = 1$$

- Theorem 1:

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Example 4

Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist

Solution

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\lim_{x\to 0^+} \frac{x}{x} = 1$$

$$\lim_{x\to 0^-} \frac{-x}{x} = -1$$

$$\therefore \lim_{x\to 0^-} \frac{|x|}{x} \neq \lim_{x\to 0^+} \frac{|x|}{x} \Rightarrow \lim_{x\to 0} \frac{|x|}{x} \text{ DNE}$$

- Theorem 2:

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

- Theorem 3: The Squeeze/Sandwich/Pinching Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

Then

$$\lim_{x\to a} g(x) = L$$

Example 5

Show that $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$

Solution

$-1 \le \sin\frac{1}{x} \le 1$ Range of sine

$-x^2 \le x^2 \cdot \sin\frac{1}{x} \le x^2$

$\lim_{x \to 0} -x^2 = 0$ and $\lim_{x \to 0} x^2 = 0$

$\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$ by sandwich theorem

Problems

- Given that $f(x) = x - 2$ and $g(x) = x^2 - 2x$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x\to 3}[f(x) + 3g(x)]$

$$= \lim_{x\to 3} f(x) + 3\lim_{x\to 3} g(x)$$

$$= \lim_{x\to 3} x-2 + 3\lim_{x\to 3} x^2-2x$$

$$= (3-2) + 3(3^2 - 2\cdot 3) = 10$$

(b) $\lim_{x\to 1}[g(x)]^3$

$$= \left[\lim_{x\to 1} g(x)\right]^3$$

$$= \left[\lim_{x\to 1} x^2 - 2x\right]^3$$

$$= (1 - 2\cdot 1)^3 = -1$$

(c) $\lim_{x\to 6}\sqrt{f(x)}$

$$= \sqrt{\lim_{x\to 6} f(x)}$$

$$= \sqrt{\lim_{x\to 6} x-2} = \sqrt{4} = 2$$

- Given that $\lim_{x\to 2} f(x) = 4$ $\lim_{x\to 2} g(x) = -2$ $\lim_{x\to 2} h(x) = 0$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x\to 2} \frac{3f(x)}{g(x)}$

$$= \frac{3 \lim_{x\to 2} f(x)}{\lim_{x\to 2} g(x)} = \frac{3 \cdot 4}{-2} = -6$$

(b) $\lim_{x\to 2} \frac{g(x)}{h(x)}$

$$= \frac{\lim_{x\to 2} g(x)}{\lim_{x\to 2} h(x)} = \frac{-2}{0} = -\infty$$

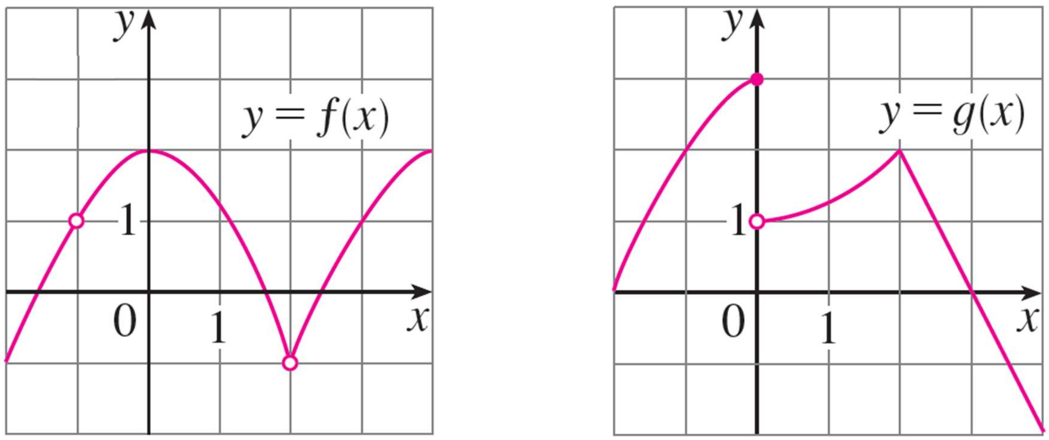
DNE أو تقدر تكتب

(c) $\lim_{x\to 2} \frac{g(x)h(x)}{f(x)}$

$$= \frac{\lim_{x\to 2} g(x) \cdot \lim_{x\to 2} h(x)}{\lim_{x\to 2} f(x)}$$

$$= \frac{-2 \cdot 0}{4} = 0$$

- The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.



(a) $\lim_{x \to 2}[f(x) + g(x)]$

$$= \lim_{x \to 2} f(x) + \lim_{x \to 2} g(x)$$

$$= -1 + 2 = 1$$

(b) $\lim_{x \to 0}[f(x) - g(x)]$

DNE

(c) $\lim_{x \to -1}[f(x)g(x)]$

$$= \lim_{x \to -1} f(x) \cdot \lim_{x \to -1} g(x) = 1 \cdot 2 = 2$$

(d) $\lim\limits_{x\to 3} \frac{f(x)}{g(x)}$

$$= \frac{\lim\limits_{x\to 3} f(x)}{\lim\limits_{x\to 3} g(x)} = \frac{1}{0} = \infty$$

(e) $\lim\limits_{x\to 2} [x^2 f(x)]$

$$= \lim_{x\to 2} x^2 \cdot \lim_{x\to 2} f(x)$$

$$= 2^2 \cdot -1 = -4$$

(f) $f(-1) + \lim\limits_{x\to -1} g(x)$

undefined

لأن $f(-1)$ غير معرّفة
undefined

- Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

(a) $\lim\limits_{x \to -1} (x^2 + x)(3x^2 + 6)$

$$= \lim_{x \to -1} (x^2 + x) \cdot \lim_{x \to -1} (3x^2 + 6) \qquad \text{law } 4$$

$$= \left[\lim_{x \to -1} x^2 + \lim_{x \to -1} x\right] \cdot \left[\lim_{x \to -1} 3x^2 + \lim_{x \to -1} 6\right] \qquad \text{law } 1$$

$$= \left[(-1)^2 + (-1)\right] \cdot \left[3 \lim_{x \to -1} x^2 + 6\right]$$

law 9 law 8 law 3 law 7

$$= 0$$

(b) $\lim\limits_{u \to -2} \sqrt{u^4 + 3u + 6}$

$$= \sqrt{\lim_{u \to -2} u^4 + 3u + 6} \qquad \text{law } 11$$

$$= \sqrt{\lim_{u \to -2} u^4 + 3 \lim_{u \to -2} u + \lim_{u \to -2} 6}$$

Law 1, law 3

$$= \sqrt{(-2)^4 + 3(-2) + 6} = \sqrt{16} = 4$$

law 9, law 8, law 7

(c) $\lim_{t \to 2} \left(\frac{t^2-2}{t^3-3t+5}\right)^2$

$$= \left(\lim_{t \to 2} \frac{t^2-2}{t^3-3t+5}\right)^2$$ law 6

$$= \frac{\lim_{t \to 2} t^2-2}{\lim_{t \to 2} t^3-3t+5}$$ law 5

$$= \frac{\lim_{t \to 2} t^2 - \lim_{t \to 2} 2}{\lim_{t \to 2} t^3 - 3\lim_{t \to 2} t + \lim_{t \to 2} 5}$$ law 1, law 2, law 3

$$= \frac{2^2 - 2}{2^3-3(2)+5}$$ law 7, law 8, law 9

$$= \frac{2}{7}$$

- Evaluate the limit, if it exists.

(a) $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$

$$= \lim_{x\to 2} \frac{(x+3)\cancel{(x-2)}}{\cancel{x-2}}$$

$$= \lim_{x\to 2} x+3 = 5$$

(b) $\lim_{t\to -3} \frac{t^2-9}{2t^2+7t+3}$

$$= \lim_{t\to -3} \frac{(t-3)\cancel{(t+3)}}{(2t+1)\cancel{(t+3)}}$$

$$= \lim_{t\to -3} \frac{t-3}{2t+1} = \frac{-6}{-5} = \frac{6}{5}$$

(c) $\lim_{h\to 0} \frac{(2+h)^3-8}{h}$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

$$= \lim_{h\to 0} \frac{\cancel{8}+12h+6h^2+h^3-\cancel{8}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(12+6h+h^2)}{\cancel{h}}$$

$$= 12$$

(d) $\lim_{x\to 3} \frac{x-3}{x^3-27}$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$$= \lim_{x\to 3} \frac{\cancel{x-3}}{\cancel{(x-3)}(x^2+3x+9)}$$

$$= \lim_{x\to 3} \frac{1}{x^2+3x+9} = \frac{1}{27}$$

(e) $\lim_{x\to -4} \frac{\sqrt{x^2+9}-5}{x+4}$

$$= \lim_{x\to -4} \frac{\sqrt{x^2+9}-5}{x+4} \cdot \frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5}$$

$$= \lim_{x\to -4} \frac{x^2+9-25}{(x+4)(\sqrt{x^2+9}+5)}$$

$$= \lim_{x\to -4} \frac{x^2-16}{(x+4)(\sqrt{x^2+9}+5)}$$

$$= \lim_{x\to -4} \frac{(x-4)\cancel{(x+4)}}{\cancel{(x+4)}(\sqrt{x^2+9}+5)}$$

$$= \lim_{x\to -4} \frac{x-4}{\sqrt{x^2+9}+5}$$

$$= \frac{-8}{10} = \frac{-4}{5}$$

- If $4x - 9 \le f(x) \le x^2 - 4x + 7$ for all $x \ge 0$, evaluate $\lim_{x \to 4} f(x)$

$$\lim_{x \to 4} 4x - 9 = 7$$

$$\lim_{x \to 4} x^2 - 4x + 7 = 7$$

$$\therefore \lim_{x \to 4} f(x) = 7$$

by sandwich/squeeze theorem

- Find $\lim_{x \to 0} x^4 \cos\frac{2}{x}$

$$-1 \le \cos\frac{2}{x} \le 1$$ Range of cosine

$$-x^4 \le x^4 \cos\frac{2}{x} \le x^4$$

$$\lim_{x \to 0} -x^4 = \lim_{x \to 0} x^4 = 0$$

$$\therefore \lim_{x \to 0} x^4 \cos\frac{2}{x} = 0$$

- **Prove that** $\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0$

$$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$$

$$e^{-1} \le e^{\sin\left(\frac{\pi}{x}\right)} \le e^{1}$$

$$\frac{1}{e} \le e^{\sin(\pi/x)} \le e$$

$$\frac{\sqrt{x}}{e} \le \sqrt{x}\, e^{\sin(\pi/x)} \le \sqrt{x}\, e$$

$$\lim_{x\to 0^+} \frac{\sqrt{x}}{e} = 0$$

$$\lim_{x\to 0^+} \sqrt{x}\, e = 0$$

$$\therefore \lim_{x\to 0^+} \sqrt{x}\, e^{\sin(\pi/x)} = 0$$

- Find the limit, if it exists. If the limit does not exist, explain why.

(a) $\lim\limits_{x\to 3}(2x+|x-3|)$

$$|x-3| = \begin{cases} x-3 & x \geq 3 \\ -(x-3) & x < 3 \end{cases}$$

$$\lim_{x\to 3^+} 2x+(x-3) = \lim_{x\to 3^+} 3x-3$$
$$= 6$$

$$\lim_{x\to 3^-} 2x-(x-3) = \lim_{x\to 3^-} x+3$$
$$= 6$$

$$\therefore \lim_{x\to 3} 2x-|x-3| = 6$$

(b) $\lim\limits_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right)$

$$\lim_{x\to 0^-} \frac{1}{x} - \frac{1}{-x}$$
$$= \lim_{x\to 0^-} \frac{1}{x} + \frac{1}{x}$$
$$= \lim_{x\to 0^-} \frac{2}{x} = \infty$$

(c) $\lim\limits_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)$

$$\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{x} = 0$$

- Let $g(x) = \frac{x^2+x-6}{|x-2|}$.

(a) Find

(i) $\lim_{x\to 2^+} g(x)$

$$\lim_{x\to 2^+} \frac{x^2+x-6}{x-2}$$

$$|x-2| = \begin{cases} x-2 & x \geq 2 \\ -(x-2) & x<2 \end{cases}$$

$$= \lim_{x\to 2^+} \frac{(x+3)\cancel{(x-2)}}{\cancel{x-2}} = 5$$

(ii) $\lim_{x\to 2^-} g(x)$

$$\lim_{x\to 2^-} \frac{x^2+x-6}{-(x-2)}$$

$$= \lim_{x\to 2^-} \frac{(x+3)\cancel{(x-2)}}{-\cancel{(x-2)}} = -5$$

(b) Does $\lim_{x\to 2} g(x)$ exist?

DNE

because $\lim_{x\to 2^+} g(x) \neq \lim_{x\to 2^-} g(x)$

- If $\lim_{x\to 1}\frac{f(x)-8}{x-1} = 10$, find $\lim_{x\to 1} f(x)$

$$\lim_{x\to 1} f(x) - 8 = 0$$

$$\lim_{x\to 1} f(x) = 8$$

- Is there a number a such that

$$\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}$$

exists? If so, find the value of *a* and the value of the limit.

$$\lim_{x\to -2} 3x^2 + ax + a + 3 = 0$$

$$3(-2)^2 - 2a + a + 3 = 0$$

$$-a = -15$$

$$a = 15$$

$$\lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2} = \lim_{x\to -2}\frac{3(x^2+5x+6)}{x^2+x-2}$$

$$= \lim_{x\to -2}\frac{3(x+3)\cancel{(x+2)}}{\cancel{(x+2)}(x-1)}$$

$$= \frac{3}{-3} = -1$$