Section 2.3 – Calculating Limits Using the Limits Laws

- Limit Laws:

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$ الجمع (1)

$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$ الطرح (2)

$$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$$ ضرب ثابت (3)

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$$ ضرب داليتن (4)

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0$$ القسمة (5)

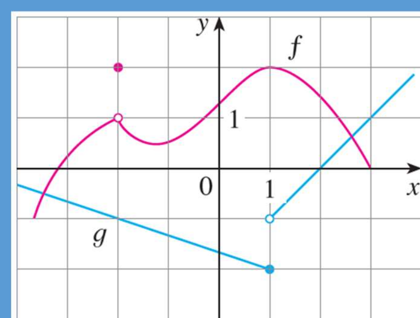
Example 1

Use the Limit Laws and the graphs of f and g in the figure to evaluate the following limits, if they exist.

(a) $\lim_{x\to -2}[f(x)+5g(x)]$

(b) $\lim_{x\to 1}[f(x)g(x)]$

(c) $\lim_{x\to 2}\frac{f(x)}{g(x)}$



Solution

(a) $\lim_{x\to -2}[f(x)+5g(x)]$

$= \lim_{x\to -2} f(x) + 5\lim_{x\to -2} g(x) = 1 + 5\cdot(-1) = -4$

(b) $\lim_{x\to 1}[f(x)\,g(x)] = \lim_{x\to 1} f(x)\ \lim_{x\to 1} g(x)$

$= \text{DNE}$, because $\lim_{x\to 1} g(x)$ DNE

(c) $\lim_{x\to 2}\frac{f(x)}{g(x)} = \text{DNE}$, because $\lim_{x\to 2} g(x) = 0$

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n \quad (6)$$

$$\lim_{x\to a} c = c \quad (7)$$

$$\lim_{x\to a} x = a \quad (8)$$

$$\lim_{x\to a} x^n = a^n \quad (9)$$

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a} \quad (10)$$

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} \quad (11)$$

Example 2

Evaluate the following limits and justify each step.

(a) $\lim_{x\to 5}(2x^2 - 3x + 4)$ (b) $\lim_{x\to 5} \frac{x^3+2x^2-1}{5-3x}$

Solution

(a) $\lim_{x\to 5} 2x^2 - \lim_{x\to 5} 3x + \lim_{x\to 5} 4$

$= 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4$

$= 2\cdot 5^2 - 3\cdot 5 + 4 = 39$

(b) $\dfrac{\lim_{x\to 5} x^3+2x^2-1}{\lim_{x\to 5} 5-3x} = \dfrac{\lim_{x\to 5} x^3 + 2\lim_{x\to 5} x^2 - \lim_{x\to 5} 1}{\lim_{x\to 5} 5 - 3\lim_{x\to 5} x}$

$= \dfrac{5^3 + 2\cdot 5^2 - 1}{5 - 3\cdot 5} = \dfrac{174}{-10} = -17.4$

Example 3

Find $\lim_{x \to 1} \frac{x^2-1}{x-1}$

Solution

$$\lim_{x \to 1} \frac{(x-1)(x+1)}{x-1}$$

التعويض المباشر لا يصلح $\frac{0}{0}$

$$= \lim_{x \to 1} x+1 = 1+1 = 2$$

لاحظ

أشهر طرق الاختصار

1- Expand ضرب

$$\lim_{h \to 0} \frac{(3+h)^2-9}{h} = \lim_{h \to 0} \frac{9+6h+h^2-9}{h} = \lim_{h \to 0} \frac{h(6+h)}{h} = \lim_{h \to 0} 6+h = 6$$

2- Factor تحليل

$$\lim_{x \to 0} \frac{x^2+x-6}{x-2} = \lim_{x \to 0} \frac{(x+3)(x-2)}{x-2} = \lim_{x \to 0} x+3 = 3$$

3- Conjugate المرافق

$$\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} = \lim_{t \to 0} \frac{(t^2+9)-9}{t^2(\sqrt{t^2+9}+3)}$$

$$= \lim_{t \to 0} \frac{t^2}{t^2(\sqrt{t^2+9}+3)} = \lim_{t \to 0} \frac{1}{\sqrt{t^2+9}+3} = \frac{1}{3+3} = \frac{1}{6}$$

مقام مشترك

4- Common denominator

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x(x+1)}\right) = \lim_{x \to 0} \frac{(x+1)-1}{x(x+1)} = \lim_{x \to 0} \frac{x}{x(x+1)} = \lim_{x \to 0} \frac{1}{x+1} = 1$$

- Theorem 1:

$$\lim_{x\to a} f(x) = L \qquad \text{if and only if} \qquad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Example 4

Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist

Solution

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\lim_{x\to 0^+} \frac{x}{x} = 1$$

$$\lim_{x\to 0^-} \frac{-x}{x} = -1$$

$$\therefore \lim_{x\to 0^-} \frac{|x|}{x} \neq \lim_{x\to 0^+} \frac{|x|}{x} \Rightarrow \lim_{x\to 0} \frac{|x|}{x} \text{ DNE}$$

- Theorem 2:

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

- Theorem 3: The Squeeze/Sandwich/Pinching Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

Then

$$\lim_{x\to a} g(x) = L$$

Example 5

Show that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$

Solution

$-1 \le \sin\frac{1}{x} \le 1$ Range of sine

$-x^2 \le x^2 \cdot \sin\frac{1}{x} \le x^2$

$\lim_{x\to 0} -x^2 = 0$ and $\lim_{x\to 0} x^2 = 0$

$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$ by sandwich theorem

Problems

- Given that $f(x) = x - 2$ and $g(x) = x^2 - 2x$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim\limits_{x \to 3}[f(x) + 3g(x)]$

(b) $\lim\limits_{x \to 1}[g(x)]^3$

(c) $\lim\limits_{x \to 6}\sqrt{f(x)}$

- Given that $\lim_{x\to 2} f(x) = 4$ $\lim_{x\to 2} g(x) = -2$ $\lim_{x\to 2} h(x) = 0$

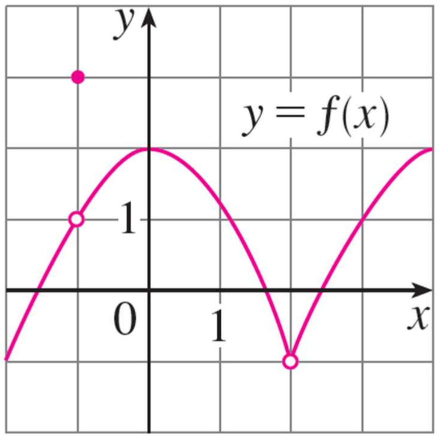
find the limits that exist. If the limit does not exist, explain why.

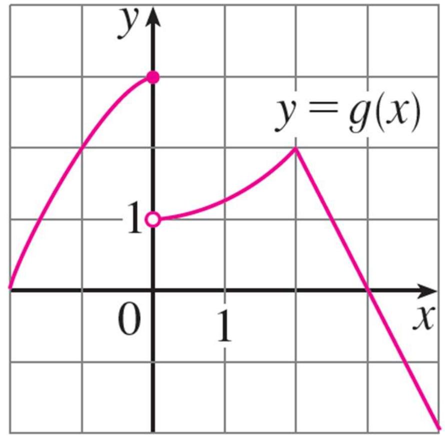
(a) $\lim_{x\to 2} \frac{3f(x)}{g(x)}$

(b) $\lim_{x\to 2} \frac{g(x)}{h(x)}$

(c) $\lim_{x\to 2} \frac{g(x)h(x)}{f(x)}$

- The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.





(a) $\lim_{x \to 2} [f(x) + g(x)]$

(b) $\lim_{x \to 0} [f(x) - g(x)]$

(c) $\lim_{x \to -1} [f(x) g(x)]$

(d) $\lim_{x \to 3} \frac{f(x)}{g(x)}$

(e) $\lim_{x \to 2} [x^2 f(x)]$

(f) $f(-1) + \lim_{x \to -1} g(x)$

- Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

(a) $\lim_{x \to -1} (x^2 + x)(3x^2 + 6)$

(b) $\lim_{u \to -2} \sqrt{u^4 + 3u + 6}$

(c) $\lim_{t\to 2}\left(\frac{t^2-2}{t^3-3t+5}\right)^2$

- Evaluate the limit, if it exists.

(a) $\lim_{x \to 2} \frac{x^2+x-6}{x-2}$

(b) $\lim_{t \to -3} \frac{t^2-9}{2t^2+7t+3}$

(c) $\lim_{h \to 0} \frac{(2+h)^3-8}{h}$

(d) $\lim_{x\to 3} \frac{x-3}{x^3-27}$

(e) $\lim_{x\to -4} \frac{\sqrt{x^2+9}-5}{x+4}$

- If $4x - 9 \le f(x) \le x^2 - 4x + 7$ for all $x \ge 0$, evaluate $\lim_{x \to 4} f(x)$

- Find $\lim_{x \to 0} x^4 \cos \frac{2}{x}$

- Prove that $\lim_{x \to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0$

- Find the limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x\to 3}(2x + |x - 3|)$

(b) $\lim_{x\to 0^-}\left(\frac{1}{x} - \frac{1}{|x|}\right)$

(c) $\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{|x|}\right)$

- Let $g(x) = \frac{x^2+x-6}{|x-2|}$.

(a) Find

(i) $\lim_{x \to 2^+} g(x)$

(ii) $\lim_{x \to 2^-} g(x)$

(b) Does $\lim_{x \to 2} g(x)$ exist?

- If $\lim_{x \to 1} \frac{f(x)-8}{x-1} = 10$, find $\lim_{x \to 1} f(x)$

- Is there a number a such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of *a* and the value of the limit.