Section 2.4 – The Precise Definition of a Limit

- Precise Definition of a Limit:

The limit of $f(x)$ as x approaches a is L

$$\lim_{x \to a} f(x) = L$$

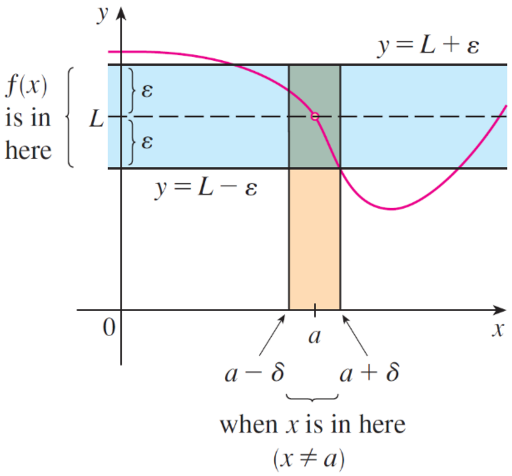
If for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

$\underbrace{|x-a|}_{\delta} \to 0 \Rightarrow \underbrace{|f(x)-L|}_{\varepsilon} \to 0$

كلما اقتربت x من a

اقتربت $f(x)$ من L



Example 1

Use the definition of limits to prove that $\lim_{x \to 3}(4x-5) = 7$

Solution

$|f(x) - L| < \varepsilon$

$|4x - 5 - 7| < \varepsilon$

$|4x - 12| < \varepsilon$

$|4(x-3)| < \varepsilon \quad \div 4$

الخطوة الأولى وضع الشكل $\boxed{x-a}$

$|x-3| < \frac{\varepsilon}{4}$

take $\delta = \frac{\varepsilon}{4}$

$|x-a| < \delta$

$|x-3| < \frac{\varepsilon}{4} \quad \times 4$

$|4x-12| < \varepsilon$

أرجع $|f(x)-L|$

$|4x-5-7| < \varepsilon$

$\therefore |f(x)-L| < \varepsilon$

Example 2

Prove the statement using the ε, δ definition of a limit.

(a) $\lim_{x\to 3}\left(1+\frac{1}{3}x\right)=2$ (b) $\lim_{x\to -2}\left(\frac{1}{2}x+3\right)=2$

Solution

(a) $f(x)=1+\frac{1}{3}x,\quad L=2,\quad a=3$

$|f(x)-L|<\varepsilon$

$\left|1+\frac{1}{3}x-2\right|<\varepsilon$

$\left|-1+\frac{1}{3}x\right|<\varepsilon \quad \times 3$

$|-3+x|<3\varepsilon$

حصي نفسي

$x-3\equiv\boxed{x-a}$

take $\delta=3\varepsilon$

$|x-a|<\delta$

$|x-3|<3\varepsilon \quad \div 3$

$\left|\frac{1}{3}x-1\right|<\varepsilon$

أرجع إلى

$1+\frac{1}{3}x-2\equiv\boxed{f(x)-L}$

$\left|1+\frac{1}{3}x-2\right|<\varepsilon$

ن $|f(x)-L|<\varepsilon$

(b) $|f(x)-L|<\varepsilon$

$\left|\frac{1}{2}x+3-2\right|<\varepsilon$

$\left|\frac{1}{2}x+1\right|<\varepsilon \quad \times 2$

$|x+2|<2\varepsilon$

take $\delta=2\varepsilon$

$|x-a|<\delta$

$|x-(-2)|<2\varepsilon$

$|x+2|<2\varepsilon \quad \div 2$

$\left|\frac{1}{2}x+1\right|<\varepsilon$

$\left|\frac{1}{2}x+3-2\right|<\varepsilon$

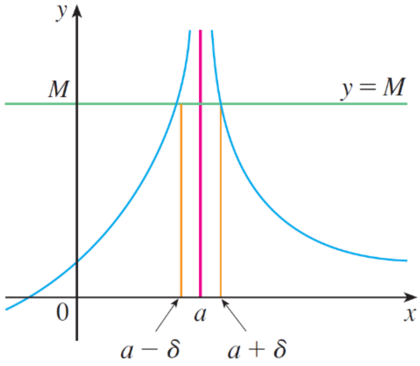
- Precise Definition of an Infinite Limit:

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if } \; 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M$$

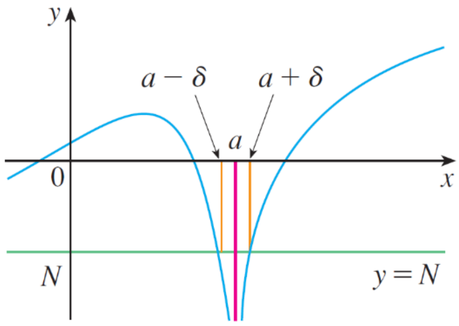
كلما اقتربت x من a
زادت قيمة $f(x)$



$$\lim_{x \to a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that

$$\text{if } \; 0 < |x - a| < \delta \quad \text{then} \quad f(x) < N$$



كلما اقتربت x من a
قلّت قيمة $f(x)$
(زادت في الاتجاه السالب)

تذكر

For $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$

a is a Vertical Asymptote.

Example 3

Find $\lim_{x\to -1} \frac{1}{x+1}$, and state the vertical asymptotes, if any.

Solution

$$\lim_{x\to -1} \frac{1}{x+1} = \frac{1}{0} = \infty$$

$x = -1$ is VA

Problems

- Prove the statement using the ε, δ definition of a limit

(a) $\lim\limits_{x \to 1}(2x+3) = 5$

$f(x) = 2x+3$, $L = 5$, $a = 1$

$|f(x) - L| < \varepsilon$

$|2x+3-5| < \varepsilon$

$|2x-2| < \varepsilon \quad \div 2$

$|x-1| < \frac{\varepsilon}{2}$

take $\delta = \frac{\varepsilon}{2}$

$|x-a| < \delta$

$|x-1| < \frac{\varepsilon}{2} \quad \times 2$

$|2x-2| < \varepsilon$

$|2x+3-5| < \varepsilon$

$\therefore |f(x) - L| < \varepsilon$

(b) $\lim\limits_{x\to 1}\frac{2+4x}{3} = 2$

$f(x) = \frac{2+4x}{3}$, $L = 2$, $a = 1$

$|f(x) - L| < \varepsilon$

$\left|\frac{2+4x}{3} - 2\right| < \varepsilon$

$\left|\frac{2}{3} + \frac{4}{3}x - 2\right| < \varepsilon$ $\qquad \frac{2}{3} - 2 = \frac{2}{3} - \frac{6}{3}$

$\left|\frac{4}{3}x - \frac{4}{3}\right| < \varepsilon$ $\qquad = \frac{-4}{3}$

$|x - 1| < \frac{3}{4}\varepsilon$

take $\delta = \frac{3}{4}\varepsilon$

$|x - a| < \delta$

$|x - 1| < \frac{3}{4}\varepsilon$ $\qquad \times \frac{4}{3}$

$\left|\frac{4}{3}x - \frac{4}{3}\right| < \varepsilon$

$\left|\frac{4}{3}x + \frac{2}{3} - 2\right| < \varepsilon$

$\left|\frac{4x+2}{3} - 2\right| < \varepsilon$

is $|f(x) - L| < \varepsilon$

(c) $\lim\limits_{x\to4}\frac{x^2-2x-8}{x-4}=6$

$$f(x)=\frac{x^2-2x-8}{x-4}\ ,\ L=6\ ,\ a=4$$

$$|f(x)-L|<\varepsilon$$

$$\left|\frac{x^2-2x-8}{x-4}-6\right|<\varepsilon$$

$$\left|\frac{x^2-2x-6}{x-4}-\frac{6(x-4)}{x-4}\right|<\varepsilon$$

$$\left|\frac{x^2-2x-6-6x+24}{x-4}\right|<\varepsilon$$

$$\left|\frac{x^2-8x+16}{x-4}\right|<\varepsilon$$

$$\left|\frac{(x-4)^2}{x-4}\right|<\varepsilon$$

$$|x-4|<\varepsilon$$

take $\delta=\varepsilon$

$$|x-a|<\delta$$

$$|x-4|<\varepsilon$$

$$\left|\frac{x^2-2x-8}{x-4}-6\right|<\varepsilon$$

$\therefore\ |f(x)-L|<\varepsilon$

(d) $\lim_{x \to a} x = a$

$|f(x) - L| < \varepsilon$

$|x - a| < \varepsilon$

take $\delta = \varepsilon$

$|x - a| < \delta$

$|x - a| < \varepsilon$

$\therefore |f(x) - L| < \varepsilon$

(e) $\lim_{x \to a} c = c$

$|f(x) - L| = 0 < \varepsilon$

الصفر أصغر دا

من أي قيمة محتملة ل ع

- Find $\lim_{x \to 1^-} \frac{x^2-6x+8}{-4x+4}$, and state the vertical asymptotes, if any.

$$\lim_{x \to 1^-} \frac{x^2-6x+8}{-4x+4} = \frac{3}{0^+} = \infty$$

$x = 1$ is VA

- Find $\lim_{x \to 1^+} \frac{x^2-6x+8}{-4x+4}$, and state the vertical asymptotes, if any.

$$\lim_{x \to 1^+} \frac{x^2-6x+8}{-4x+4} = \frac{3}{0^-} = -\infty$$

$x = 1$ is VA