

Section 2.4 – The Precise Definition of a Limit

- Precise Definition of a Limit:

The limit of $f(x)$ as x approaches a is L

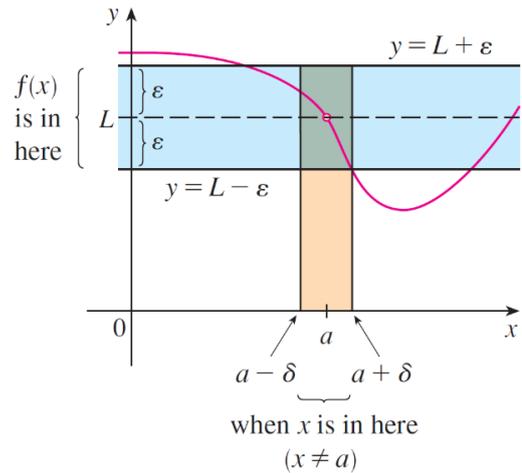
$$\lim_{x \rightarrow a} f(x) = L$$

If for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

$$\underbrace{|x - a|}_{\delta} \rightarrow 0 \Rightarrow \underbrace{|f(x) - L|}_{\varepsilon} \rightarrow 0$$

كل اقتربت x من a
اقتربت $f(x)$ من L



Example 1

Use the definition of limits to prove that $\lim_{x \rightarrow 3} (4x - 5) = 7$

Solution

$$|f(x) - L| < \varepsilon$$

$$|4x - 5 - 7| < \varepsilon$$

$$|4x - 12| < \varepsilon$$

$$|4(x - 3)| < \varepsilon \quad \div 4$$

$$|x - 3| < \frac{\varepsilon}{4}$$

$$\text{take } \delta = \frac{\varepsilon}{4}$$

$$|x - a| < \delta$$

$$|x - 3| < \frac{\varepsilon}{4} \quad \times 4$$

$$|4x - 12| < \varepsilon$$

$$|4x - 5 - 7| < \varepsilon$$

$$\therefore |f(x) - L| < \varepsilon$$

الخطوة
الأولى
وهي
تأخذ
 $x - a$

Example 2

Prove the statement using the ε, δ definition of a limit.

(a) $\lim_{x \rightarrow 3} (1 + \frac{1}{3}x) = 2$

(b) $\lim_{x \rightarrow -2} (\frac{1}{2}x + 3) = 2$

Solution

(a) $f(x) = 1 + \frac{1}{3}x, L = 2, a = 3$

$$|f(x) - L| < \varepsilon$$

$$|1 + \frac{1}{3}x - 2| < \varepsilon$$

$$|-1 + \frac{1}{3}x| < \varepsilon \quad \times 3$$

$$|x - 3| < 3\varepsilon$$

take $\delta = 3\varepsilon$

$$|x - a| < \delta$$

$$|x - 3| < 3\varepsilon \quad \div 3$$

$$|\frac{1}{3}x - 1| < \varepsilon$$

$$|1 + \frac{1}{3}x - 2| < \varepsilon$$

$$\therefore |f(x) - L| < \varepsilon$$

فهمي

$$x - 3 \equiv |x - a|$$

فهمي

$$1 + \frac{1}{3}x - 2 \equiv |f(x) - L|$$

(b) $|f(x) - L| < \varepsilon$

$$|\frac{1}{2}x + 3 - 2| < \varepsilon$$

$$|\frac{1}{2}x + 1| < \varepsilon \quad \times 2$$

$$|x + 2| < 2\varepsilon$$

take $\delta = 2\varepsilon$

$$|x - a| < \delta$$

$$|x - (-2)| < 2\varepsilon$$

$$|x + 2| < 2\varepsilon \quad \div 2$$

$$|\frac{1}{2}x + 1| < \varepsilon$$

$$|\frac{1}{2}x + 3 - 2| < \varepsilon$$

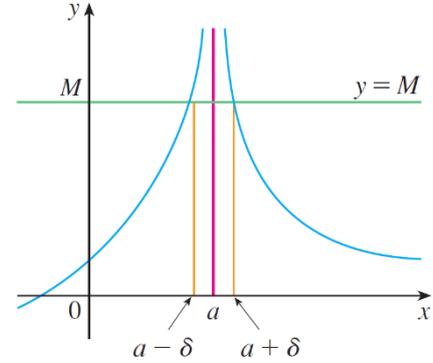
- Precise Definition of an Infinite Limit:

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } f(x) > M$$

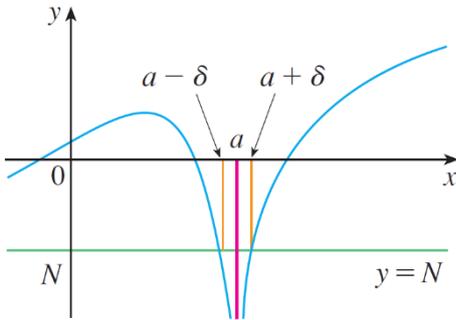
كلما اعتبرت x من a
زادت قيمة $f(x)$



$$\lim_{x \rightarrow a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } f(x) < N$$



كلما اعتبرت x من a
قلت قيمة $f(x)$
(زادت في الاتجاه السالب)

تذكر

$$\text{For } \lim_{x \rightarrow a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = \pm\infty$$

a is a Vertical Asymptote.

Example 3

Find $\lim_{x \rightarrow -1} \frac{1}{x+1}$, and state the vertical asymptotes, if any.

Solution

$$\lim_{x \rightarrow -1} \frac{1}{x+1} = \frac{1}{0} = \infty$$

$x = -1$ is VA

Problems

- Prove the statement using the ε, δ definition of a limit

(a) $\lim_{x \rightarrow 1} (2x + 3) = 5$

$$(b) \lim_{x \rightarrow 1} \frac{2+4x}{3} = 2$$

$$(c) \lim_{x \rightarrow 4} \frac{x^2 - 2x - 8}{x - 4} = 6$$

(d) $\lim_{x \rightarrow a} x = a$

(e) $\lim_{x \rightarrow a} c = c$

- Find $\lim_{x \rightarrow 1^-} \frac{x^2 - 6x + 8}{-4x + 4}$, and state the vertical asymptotes, if any.

- Find $\lim_{x \rightarrow 1^+} \frac{x^2 - 6x + 8}{-4x + 4}$, and state the vertical asymptotes, if any.