Section 2.6 – Limits at Infinity; Horizontal Asymptotes

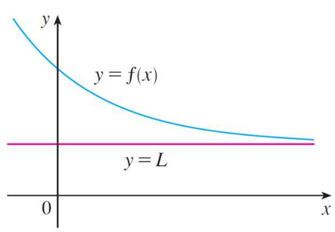
- Limit at infinity:

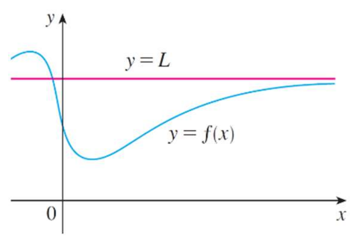
Let f be a function defined on some interval (a, ∞). Then

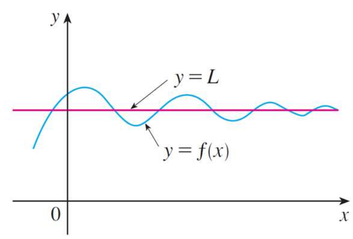
$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

كلما زادت قيمة x تقترب f(x) من قيمة معينة L



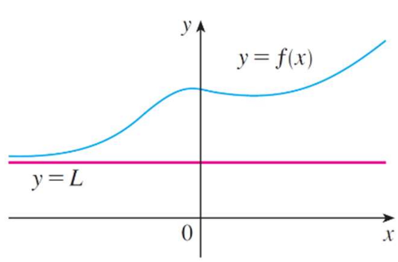


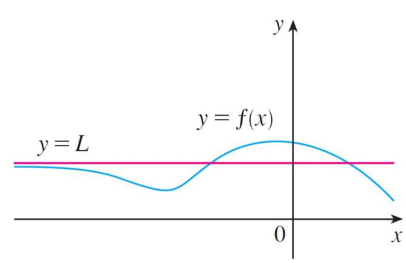


- Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.





كلما زادت قيمة x في الاتجاه السالب
اقتربت f(x) من قيمة معينة L

لاحظ

$\lim_{x\to\infty} \frac{1}{x} = 0^+$

$\lim_{x\to-\infty} \frac{1}{x} = 0^-$

$\lim_{x\to\infty} a^x = \infty, \quad a > 1$

$\lim_{x\to\infty} a^{-x} = \lim_{x\to\infty} \frac{1}{a^x} = 0^+, \quad a > 1$

$\lim_{x\to\infty} \frac{x^n}{a^x} = 0^+, \quad a > 1$

$\lim_{x\to\infty} a^x = 0 \quad , \quad 0 < a < 1$ صحي تقسيها

$\lim_{x\to\infty} \frac{1}{x^r} = 0^+, \quad r \in \mathbb{Q}^+$

$\lim_{x\to-\infty} \frac{1}{x^r} = 0, \quad r \in \mathbb{Q}^+$

الأعداد النسبية

Example 1

Evaluate

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

Solution

$$\lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}}{\frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2}}$$

نقسم على x أكبر أس في المقام

$$= \lim_{x\to\infty} \frac{3 - \cancel{\frac{1}{x}} - \cancel{\frac{2}{x^2}}}{5 + \cancel{\frac{4}{x}} + \cancel{\frac{1}{x^2}}}$$

$$= \frac{3}{5}$$

Example 2

Evaluate

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}$$

Solution

$$\lim_{x\to\infty}\frac{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}{\frac{3x}{x}-\frac{5}{x}} = \lim_{x\to\infty}\frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{3x}{x}-\frac{5}{x}}$$

$$= \lim_{x\to\infty}\frac{\sqrt{2+\cancel{\frac{1}{x^2}}}}{3-\cancel{\frac{5}{x}}}$$

$$= \frac{\sqrt{2}}{3}$$

Example 3

Compute $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$

Solution

$$\lim_{x\to\infty}\sqrt{x^2+1}-x \cdot \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty}\frac{\cancel{x^2}+1-\cancel{x^2}}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x} = \frac{1}{\infty} = 0$$

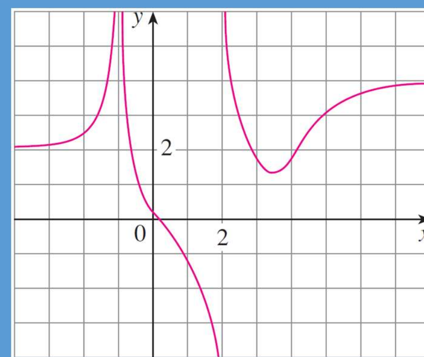
- Horizontal Asymptote:

The line $y = L$ is called **horizontal asymptote** of the curve $y = f(x)$ if either

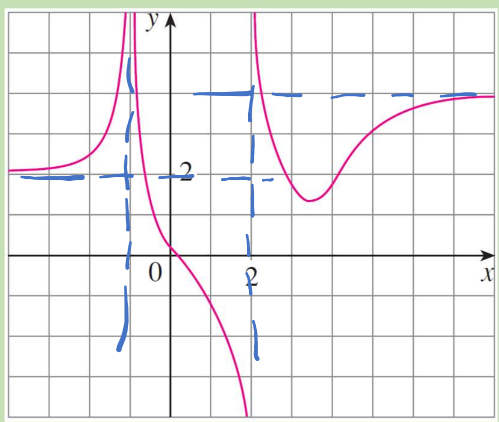
$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

Example 4

Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown



Solution



$\lim_{x\to -1} f(x) = \infty$, $\lim_{x\to 2^+} f(x) = \infty$, $\lim_{x\to 2^-} f(x) = -\infty$

therefore $x = -1$ and $x = 2$ are VA

$\lim_{x\to \infty} f(x) = 4$ and $\lim_{x\to -\infty} f(x) = 2$

therefore $y = 4$ and $y = 2$ are HA

Example 5

Find the horizontal and vertical asymptotes of the graph of the function

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}$$

Solution

$$\lim_{x\to\infty}\frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{3x}{x}-\frac{5}{x}} =$$

$$= \lim_{x\to\infty}\frac{\sqrt{2+\cancel{\frac{1}{x^2}}}}{3-\cancel{\frac{5}{x}}} = \frac{\sqrt{2}}{3} \Rightarrow y=\frac{\sqrt{2}}{3} \text{ is HA}$$

$$\lim_{x\to-\infty}\frac{-\sqrt{\frac{2x^2}{x^2}+\cancel{\frac{1}{x^2}}}}{3-\cancel{\frac{5}{x}}} \qquad \sqrt{x^2}=|x|=\begin{cases}x & x\geq 0\\ -x & x<0\end{cases}$$

$$= -\frac{\sqrt{2}}{3} \Rightarrow y=-\frac{\sqrt{2}}{3} \text{ is HA}$$

$$3x-5=0$$

$$3x=5$$

$$x=\frac{5}{3}$$

since numerator $\neq 0$ at $x=\frac{5}{3}$

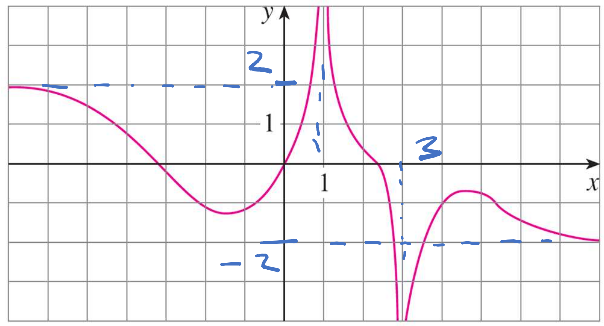
therefore $x=\frac{5}{3}$ is VA

Problems

- For the function f whose graph is given, state the following.

(a) $\lim_{x\to\infty} f(x)$

$= -2$



(b) $\lim_{x\to-\infty} f(x) = 2$

(c) $\lim_{x\to 1} f(x) = \infty$

(d) $\lim_{x\to 3} f(x) = -\infty$

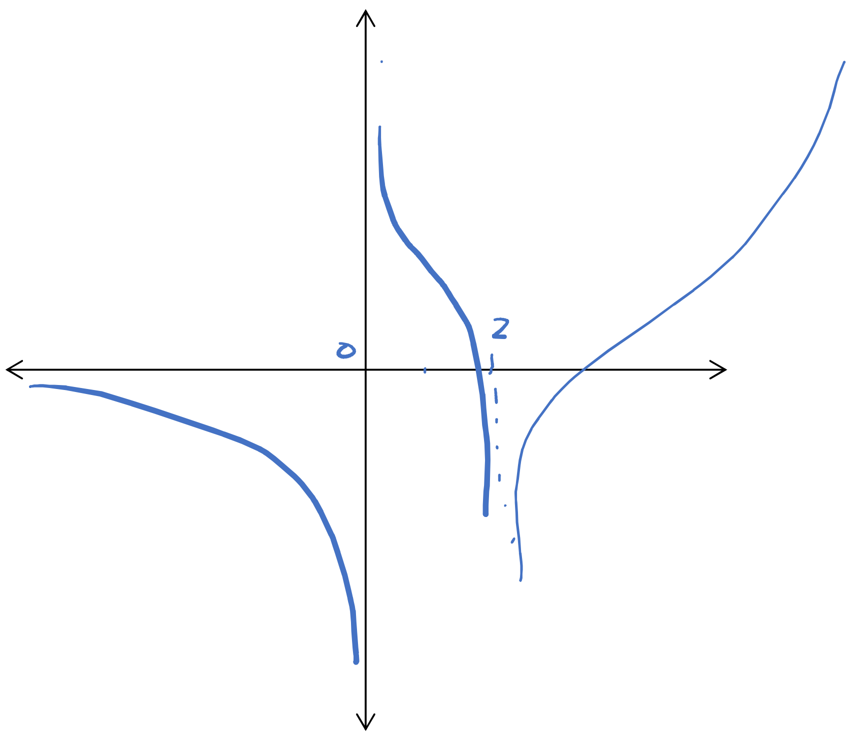
(e) The equations of the asymptotes

HA: $y = 2$, $y = -2$

VA: $x = 1$, $x = 3$

- Sketch the graph of an example of a function f that satisfies all of the given conditions.

$\lim_{x\to 2} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 0$,
$\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = -\infty$



- Find the limit or show that it doesn't exist

(a) $\lim_{x\to\infty} \frac{3x+5}{x-4}$

$$\lim_{x\to\infty} \frac{\frac{3x}{x}+\frac{5}{x}}{\frac{x}{x}-\frac{4}{x}}$$

$$= \lim_{x\to\infty} \frac{3+\cancel{\frac{5}{x}}^{0}}{1-\cancel{\frac{4}{x}}^{0}} = \frac{3}{1} = 3$$

(b) $\lim_{y\to\infty} \frac{2-3y^2}{5y^2+4y}$

$$\lim_{y\to\infty} \frac{\frac{2}{y^2}-\frac{3y^2}{y^2}}{\frac{5y^2}{y^2}+\frac{4y}{y^2}}$$

$$= \lim_{y\to\infty} \frac{\cancel{\frac{2}{y^2}}^{0}-3}{5+\cancel{\frac{4}{y}}^{0}} = \frac{-3}{5}$$

(c) $\lim\limits_{t\to\infty}\frac{t-t\sqrt{t}}{2t^{3/2}+3t-5}$

$$\lim_{t\to\infty}\frac{\dfrac{t-t\cdot t^{1/2}}{t^{3/2}}}{\dfrac{2t^{3/2}+3t-5}{t^{3/2}}}$$

$t\cdot t^{1/2}=t^{1+1/2}$

$$=\lim_{t\to\infty}\frac{\dfrac{t}{t^{3/2}}-\dfrac{t^{3/2}}{t^{3/2}}}{2\dfrac{t^{3/2}}{t^{3/2}}+\dfrac{3t}{t^{3/2}}-\dfrac{5}{t^{3/2}}}$$

$\frac{t}{t^{3/2}}=t^{1-3/2}=t^{-1/2}$

$$=\lim_{t\to\infty}\frac{\cancel{\dfrac{1}{t^{1/2}}}^{0}-1}{2+\cancel{\dfrac{3}{t^{1/2}}}^{0}-\cancel{\dfrac{5}{t^{3/2}}}^{0}}=\frac{-1}{2}$$

(d) $\lim\limits_{x\to-\infty}\frac{\sqrt{9x^6-x}}{x^3+1}$

$$\lim_{x\to-\infty}\frac{\dfrac{\sqrt{9x^6-x}}{-\sqrt{x^6}}}{\dfrac{x^3+1}{x^3}}$$

$\sqrt{x^6}=|x^3|=\begin{cases}x^3 & x\geq 0\\ -x^3 & x<0\end{cases}$

$$=\lim_{x\to-\infty}\frac{-\sqrt{\dfrac{9x^6}{x^6}-\dfrac{x}{x^6}}}{\dfrac{x^3}{x^3}+\dfrac{1}{x^3}}$$

$$=\lim_{x\to-\infty}\frac{-\sqrt{9-\cancel{\dfrac{1}{x^5}}^{0}}}{1+\cancel{\dfrac{1}{x^3}}^{0}}=\frac{-\sqrt{9}}{1}=-3$$

(e) $\lim\limits_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)$

$$\lim_{x\to-\infty} \sqrt{4x^2+3x}+2x \cdot \frac{\sqrt{4x^2+3x}-2x}{\sqrt{4x^2+3x}-2x}$$

$$= \lim_{x\to-\infty} \frac{\cancel{4x^2}+3x-\cancel{4x^2}}{\sqrt{4x^2+3x}-2x}$$

$$= \lim_{x\to-\infty} \frac{\frac{3x}{x}}{\frac{\sqrt{4x^2+3x}-2x}{x}}$$

$$= \lim_{x\to-\infty} \frac{\frac{3x}{x}}{\frac{\sqrt{4x^2+3x}}{-\sqrt{x^2}}-\frac{2x}{x}}$$

$$= \lim_{x\to-\infty} \frac{\frac{3x}{x}}{-\sqrt{\frac{4x^2}{x^2}+\frac{3x}{x^2}}-\frac{2x}{x}}$$

$$= \lim_{x\to-\infty} \frac{3}{-\sqrt{4+\cancel{\frac{3}{x}}_{\,0}}-2}$$

$$= \frac{3}{-2-2} = \frac{3}{-4} = -\frac{3}{4}$$

(f) $\lim\limits_{x\to-\infty} \frac{1+x^6}{x^4+1}$

$$\lim_{x\to-\infty} \frac{\frac{1}{x^4}+\frac{x^6}{x^4}}{\frac{x^4}{x^4}+\frac{1}{x^4}}$$

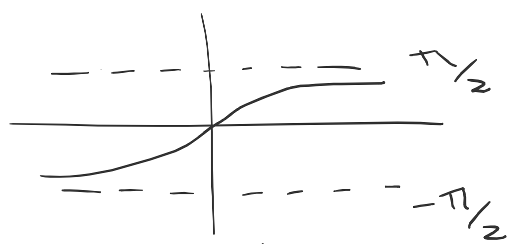
$$= \lim_{x\to-\infty} \frac{\cancel{\frac{1}{x^4}}^{0}+x^2}{1+\cancel{\frac{1}{x^4}}^{0}}$$

$$= \frac{(-\infty)^2}{1} = \infty$$

(g) $\lim\limits_{x\to\infty} \arctan(e^x)$

$$\lim_{x\to\infty} \tan^{-1}(e^x)$$

$$= \tan^{-1}(\infty) = \frac{\pi}{2}$$



$y = \tan^{-1} x$

(h) $\lim_{x\to\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}$

$$\lim_{x\to\infty} \frac{\frac{e^{3x}}{e^{3x}} - \frac{e^{-3x}}{e^{3x}}}{\frac{e^{3x}}{e^{3x}} + \frac{e^{-3x}}{e^{3x}}}$$

$$\frac{e^{-3x}}{e^{3x}} = e^{-3x-3x} = e^{-6x}$$

$$= \lim_{x\to\infty} \frac{1 - \cancel{\frac{1}{e^{6x}}}^{\,0}}{1 + \cancel{\frac{1}{e^{6x}}}^{\,0}} = 1$$

(i) $\lim_{x\to\infty} (e^{-2x}\cos x)$

$$-1 \le \cos x \le 1$$

$$-e^{-2x} \le e^{-2x}\cos x \le e^{-2x}$$

$$\lim_{x\to\infty} -e^{-2x} = \lim_{x\to\infty} -\frac{1}{e^{2x}} = 0$$

$$\lim_{x\to\infty} e^{-2x} = \lim_{x\to\infty} \frac{1}{e^{2x}} = 0$$

$$\therefore \lim_{x\to\infty} e^{-2x}\cos x = 0$$ by sandwich theorem

- Find the horizontal and vertical asymptotes of each curve.

(a) $y = \frac{5+4x}{x+3}$

$$\lim_{x\to\infty} \frac{\frac{5}{x} + \frac{4x}{x}}{\frac{x}{x} + \frac{3}{x}} = \lim_{x\to\infty} \frac{\cancel{\frac{5}{x}}^{0} + 4}{1 + \cancel{\frac{3}{x}}^{0}} = 4$$

$$\lim_{x\to-\infty} \frac{\cancel{\frac{5}{x}}^{0} + 4}{1 + \cancel{\frac{3}{x}}^{0}} = 4$$

HA: $y = 4$

$x + 3 = 0 \Rightarrow x = -3$

VA: $x = -3$

(b) $y = \frac{x^3 - x}{x^2 - 6x + 5}$

$$\lim_{x \to \infty} \frac{\frac{x^3}{x^2} - \frac{x}{x^2}}{\frac{x^2}{x^2} - \frac{6x}{x^2} + \frac{5}{x^2}}$$

$$= \lim_{x \to \infty} \frac{x - \cancel{\frac{1}{x}}^{\,0}}{1 - \underbrace{\cancel{\frac{6}{x}}}_{0} + \underbrace{\cancel{\frac{5}{x^2}}}_{0}} = \frac{\infty}{1} = \infty$$

$$\lim_{x \to -\infty} \frac{x - \cancel{\frac{1}{x}}^{\,0}}{1 - \underbrace{\cancel{\frac{6}{x}}}_{0} + \cancel{\frac{5}{x^2}}^{\,0}} = \frac{-\infty}{1} = -\infty$$

No HA

VA

هي أصفار المقام

والدالة في أبسط صورة

$$y = \frac{x(x^2 - 1)}{(x-5)(x-1)}$$

$$= \frac{x(x+1)\cancel{(x-1)}}{(x-5)\cancel{(x-1)}} = \frac{x(x+1)}{x-5}$$

$$x - 5 = 0 \Rightarrow x = 5$$

VA: $x = 5$