Section 2.6 – Limits at Infinity; Horizontal Asymptotes

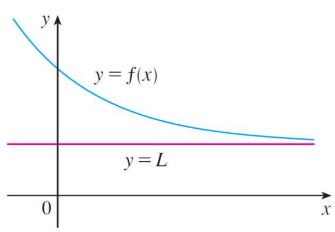
- Limit at infinity:

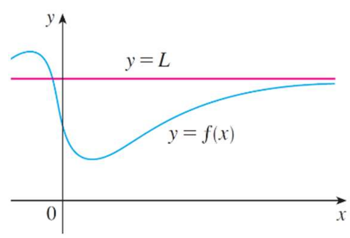
Let f be a function defined on some interval (a, ∞). Then

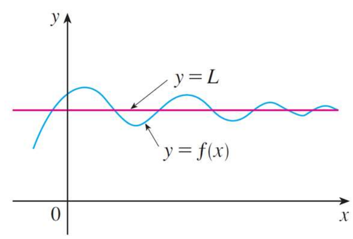
$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

كلما زادت قيمة x تقترب f(x) من قيمة معينة لِـ L



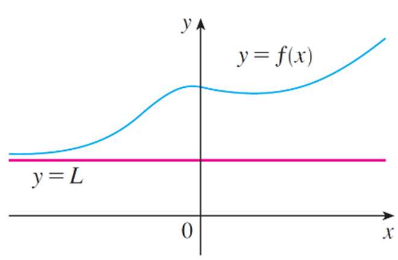


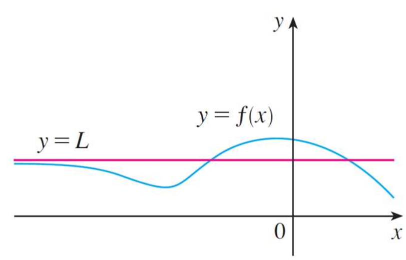


- Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.





كلما زادت قيمة x في الاتجاه السالب
اقتربت f(x) من قيمة معينة L

لاحظ

$\lim_{x\to\infty} \frac{1}{x} = 0^+$

$\lim_{x\to-\infty} \frac{1}{x} = 0^-$

$\lim_{x\to\infty} a^x = \infty, \quad a > 1$

$\lim_{x\to\infty} a^{-x} = \lim_{x\to\infty} \frac{1}{a^x} = 0^+, \quad a > 1$

$\lim_{x\to\infty} a^x = 0 \, , \, 0 < a < 1$ صح تقسيمها

$\lim_{x\to\infty} \frac{x^n}{a^x} = 0^+, \quad a > 1$

$\lim_{x\to\infty} \frac{1}{x^r} = 0^+, \quad r \in \mathbb{Q}^+$ ← الأعداد النسبية

$\lim_{x\to-\infty} \frac{1}{x^r} = 0, \quad r \in \mathbb{Q}^+$

Example 1

Evaluate

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

Solution

$$\lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}}{\frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2}}$$

نقسم على x أكبر أس في المقام

$$= \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

$$= \frac{3}{5}$$

Example 2

Evaluate

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}$$

Solution

$$\lim_{x\to\infty}\frac{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}{\frac{3x}{x}-\frac{5}{x}}=\lim_{x\to\infty}\frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{3x}{x}-\frac{5}{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}}$$

$$=\frac{\sqrt{2}}{3}$$

Example 3

Compute $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$

Solution

$$\lim_{x\to\infty}\sqrt{x^2+1}-x\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$=\lim_{x\to\infty}\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=\frac{1}{\infty}=0$$

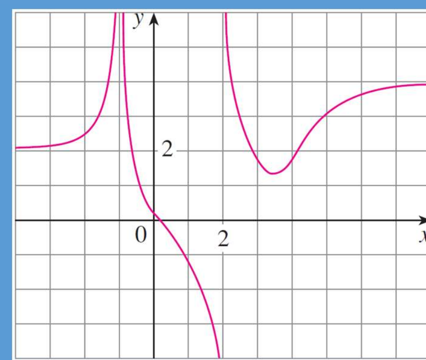
- Horizontal Asymptote:

The line $y = L$ is called **horizontal asymptote** of the curve $y = f(x)$ if either

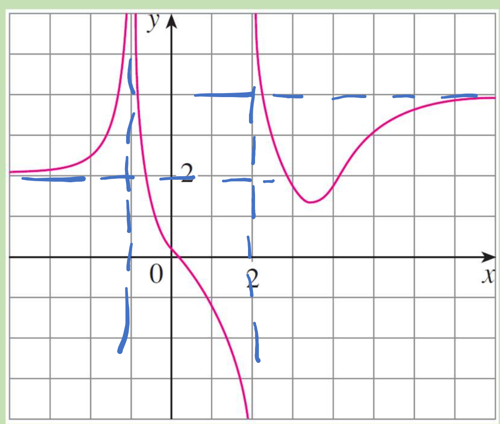
$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

Example 4

Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown



Solution



$\lim_{x\to -1} f(x) = \infty$, $\lim_{x\to 2^+} f(x) = \infty$, $\lim_{x\to 2^-} f(x) = -\infty$

therefore $x = -1$ and $x = 2$ are VA

$\lim_{x\to\infty} f(x) = 4$ and $\lim_{x\to-\infty} f(x) = 2$

therefore $y = 4$ and $y = 2$ are HA

Example 5

Find the horizontal and vertical asymptotes of the graph of the function

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

Solution

$$\lim_{x\to\infty} \frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{3x}{x}-\frac{5}{x}} =$$

$$= \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{2}}{3} \Rightarrow y = \frac{\sqrt{2}}{3} \text{ is HA}$$

$$\lim_{x\to-\infty} \frac{-\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{3-\frac{5}{x}} \qquad \sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$= -\frac{\sqrt{2}}{3} \Rightarrow y = -\frac{\sqrt{2}}{3} \text{ is HA}$$

$$3x - 5 = 0$$

$$3x = 5$$

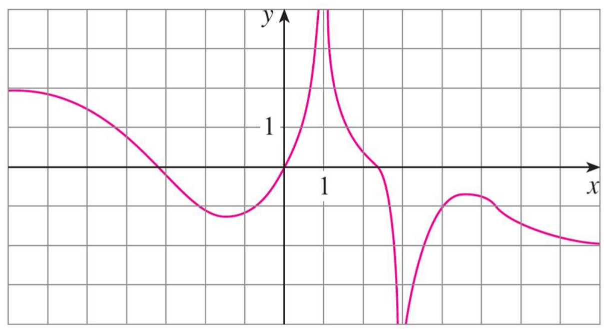
$$x = \frac{5}{3}$$

since numerator $\neq 0$ at $x = \frac{5}{3}$

therefore $x = \frac{5}{3}$ is VA

Problems

- For the function f whose graph is given, state the following.



(a) $\lim_{x \to \infty} f(x)$

(b) $\lim_{x \to -\infty} f(x)$

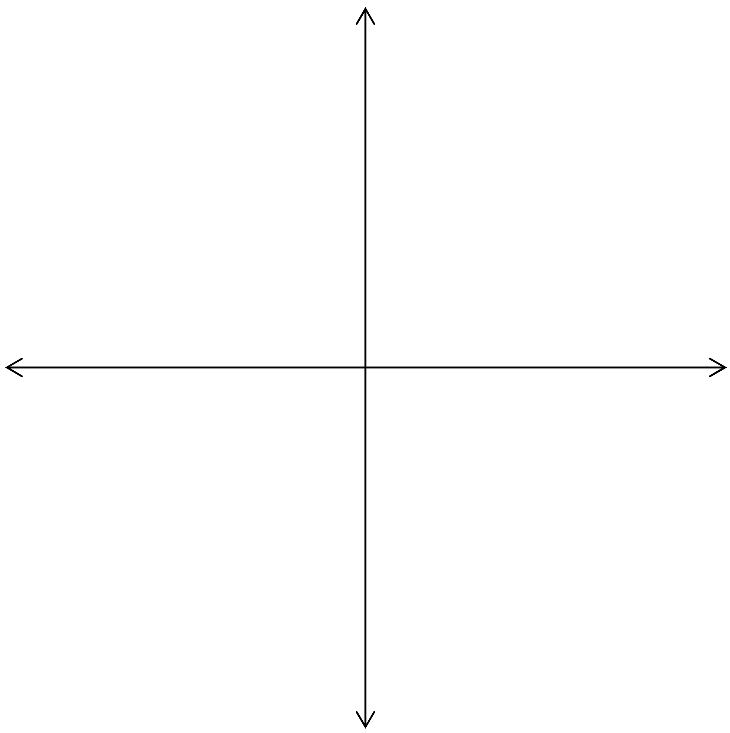
(c) $\lim_{x \to 1} f(x)$

(d) $\lim_{x \to 3} f(x)$

(e) The equations of the asymptotes

- Sketch the graph of an example of a function f that satisfies all of the given conditions.

$\lim_{x \to 2} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to -\infty} f(x) = 0$,
$\lim_{x \to 0^+} f(x) = \infty$, $\lim_{x \to 0^-} f(x) = -\infty$



- Find the limit or show that it doesn't exist

(a) $\lim_{x\to\infty} \frac{3x+5}{x-4}$

(b) $\lim_{y\to\infty} \frac{2-3y^2}{5y^2+4y}$

(c) $\lim_{t \to \infty} \frac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5}$

(d) $\lim_{x \to -\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$

(e) $\lim_{x \to -\infty} \left(\sqrt{4x^2 + 3x} + 2x \right)$

(f) $\lim_{x \to -\infty} \frac{1+x^6}{x^4+1}$

(g) $\lim_{x \to \infty} \arctan(e^x)$

(h) $\lim_{x\to\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}$

(i) $\lim_{x\to\infty} (e^{-2x}\cos x)$

- Find the horizontal and vertical asymptotes of each curve.

(a) $y = \frac{5+4x}{x+3}$

(b) $y = \frac{x^3 - x}{x^2 - 6x + 5}$