Section 2.5 – Continuity

- Continuity:

A function f is **continuous at a number a** if

$$\lim_{x\to a} f(x) = f(a)$$

① $f(a)$ defined

② $\lim_{x\to a} f(x)$ exists

③ $\lim_{x\to a} f(x) = f(a)$

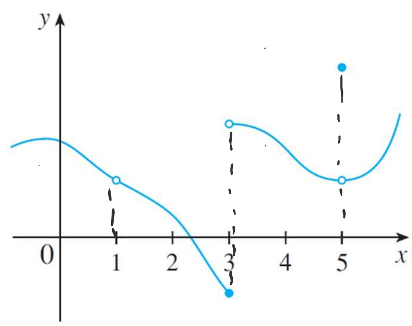
} شرط واحد يتضمن 3 شروط

discontinuous at:

$x = 1 \Rightarrow f(1)$ undefined

$x = 3 \Rightarrow \lim_{x\to 3} f(x)$ DNE

$x = 5 \Rightarrow \lim_{x\to 5} f(x) \neq f(5)$



Example 1

Where are each of the following functions discontinuous?

(a) $f(x) = \frac{x^2-x-2}{x-2}$

(b) $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Solution

(a) $f(2)$ not defined

$\therefore f(x)$ is discontinuous at $x=2$

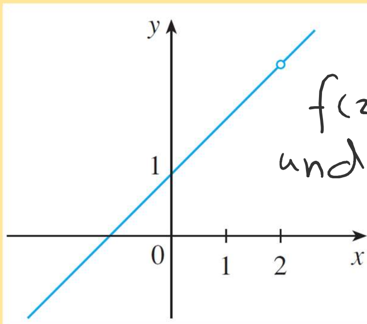
(b) $\lim_{x\to 0} \frac{1}{x^2}$ DNE

$\therefore f(x)$ is discontinuous at $x=0$

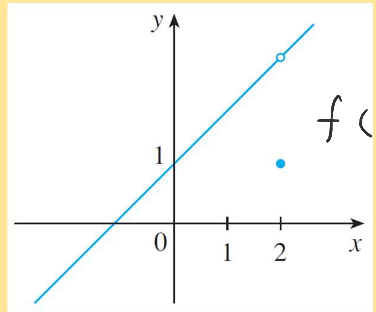
لاحظ

يوجد 3 أنواع لل discontinuity

1. Removable Discontinuity



$f(2)$ undefined

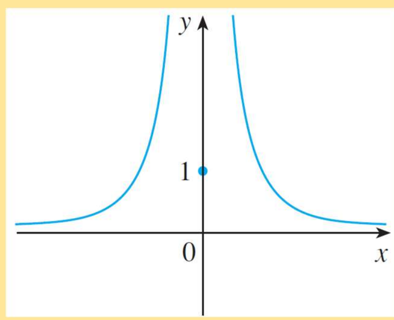


$f(2) \neq \lim_{x\to 2} f(x)$

$$f(x) = \frac{x^2-x-2}{x-2}$$

$$f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if} \quad x \neq 2 \\ 1 & \text{if} \quad x = 2 \end{cases}$$

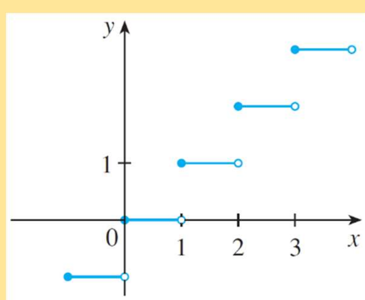
2. Infinite Discontinuity



$\lim_{x\to 0} f(x) = \infty$

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

3. Jump Discontinuity



$\lim_{x\to a} f(x)$ DNE

$$f(x) = [\![x]\!]$$

- One-side continuity: اتصال من ناحية واحدة

A function f is **continuous from the right at a number a** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \to a^-} f(x) = f(a)$$

- Continuity on an interval:

A function f is **continuous on an interval (a, b)** if it is continuous at every number in the interval (on every point $x \in (a, b)$).

الدالة f متصلة في الفترة (a,b) إذا كان متصلة

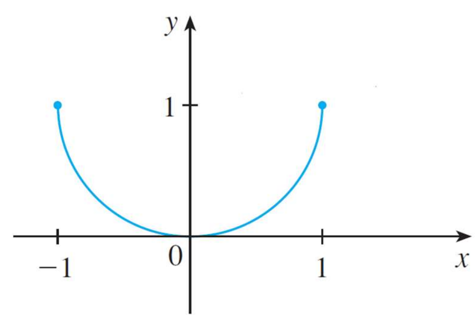
عند كل القيم بين $x = a$ و $x = b$

$f(x)$ continuous on the interval $[-1,1]$

فترة مغلقة لأنها متصلة عند

$x = -1$ من ناحية اليمين

وعند $x = 1$ من ناحية اليسار



Example 2

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1,1]$.

Solution

for $-1 \leq a \leq 1$

$$\lim_{x \to a} 1 - \sqrt{1 - x^2} = 1 - \lim_{x \to a} \sqrt{1 - x^2}$$

$$= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} = 1 - \sqrt{1 - a^2}$$

also $f(a) = 1 - \sqrt{1 - a^2}$

$\therefore f(a) = \lim_{x \to a} f(x) \Rightarrow f(x)$ is continuous for $x = [-1,1]$

- Theorem:

If f and g are continuous at a, and c is constant, then the following functions are also continuous at a:

1. $f+g$
2. $f-g$
3. cf
4. fg
5. $\frac{f}{g}$ if $g(a) \neq 0$

- Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

الحدوديات متصلة عند كل الأعداد الحقيقية

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

الدوال الكسرية متصلة عند كل قيم x المعرفة عندها

Example 3

Find $\lim_{x \to -2} \frac{x^3+2x^2-1}{5-3x}$

Solution

$f(x) = \dfrac{x^3+2x^2-1}{5-3x}$ is rational function

$\therefore f(x)$ is continuous on its domain

$\mathbb{R} / \{5/3\}$ صفر المقام

$$\lim_{x \to -2} f(x) = f(-2) = \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}$$

$$= -\frac{1}{11}$$

- Theorem:

The following types of functions are continuous at every number in their domains:

الدوال التالية متصلة عند كل القيم في مجالها:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

Example 4

Where is the function $f(x) = \frac{\ln x + \tan^{-1} x}{x^2-1}$ continuous?

Solution

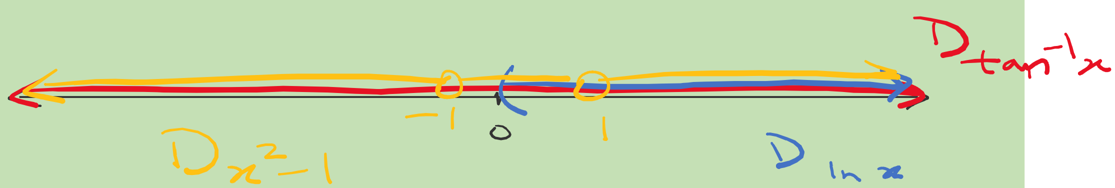
$$D_{\ln x} = (0, \infty)$$

$$D_{\tan^{-1} x} = (-\infty, \infty)$$

$$D_{x^2-1} = \mathbb{R} / \{\pm 1\}$$ أصفار المقام

$$D_f = D_{\ln x} \cap D_{\tan^{-1} x} \cap D_{x^2-1}$$

$$= (0,1) \cup (1, \infty)$$



$f(x)$ is continuous on its domain

$$(0,1) \cup (1,\infty) \equiv \mathbb{R}^+ / \{1\}$$

- Theorem:

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

نهاية الدالة المركبة

Example 5

Evaluate $\lim_{x \to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$.

Solution

$$\lim_{x \to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)$$

$$= \sin^{-1}\left(\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x}\right)$$

$$= \sin^{-1}\left(\lim_{x \to 1} \frac{\cancel{1-\sqrt{x}}}{\cancel{(1-\sqrt{x})}(1+\sqrt{x})}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Example 6

Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$ (b) $F(x) = \ln(1 + \cos x)$

Solution

(a) $h(x) = f(g(x))$

$g(x) = x^2$ is continuous on $\mathbb{R}$

$f(x) = \sin x$ is continuous on $\mathbb{R}$

$\therefore$ $h(x)$ is continuous on $\mathbb{R}$

(b) $H(x) = 1 + \cos x$ is continuous on $\mathbb{R}$

$G(x) = \ln(1 + \cos x)$

is continuous on $1 + \cos x > 0$

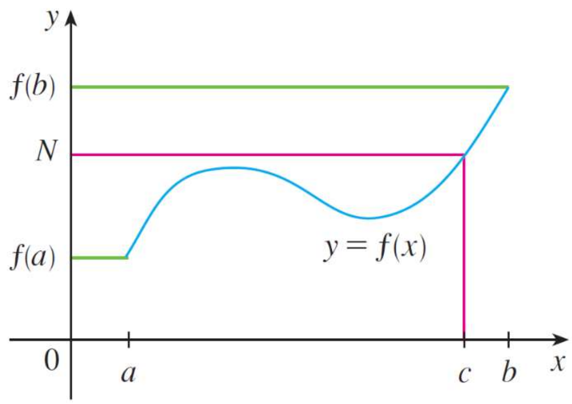
$\cos x > -1 \Rightarrow \cos x \neq -1$

$x \neq \pi + 2k\pi \, , \, k \in \mathbb{Z}$

- The Intermediate Value Theorem: مهم

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$

$$f(a) < N < f(b) \quad \Rightarrow \quad \exists c \in (a, b) \text{ such that } f(c) = N$$



Example 7

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

Solution

$f(1) = 4(1)^3 - 6(1)^2 + 3 \cdot 1 - 2 = -1$

$f(2) = 4(2)^3 - 6(2)^2 + 3 \cdot 2 - 2 = 12$

for $1 < c < 2$

$f(1) < f(c) < f(2)$

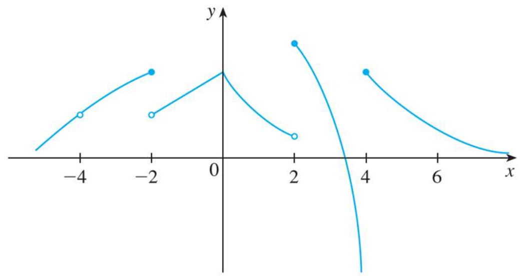
$-1 < f(c) < 12$

since $f(x)$ is continuous on $\mathbb{R}$

$\therefore f(c) = 0 \Rightarrow c$ is a root for $f(x)$

Problems

- From the graph of f, state the numbers at which f is discontinuous and explain why.



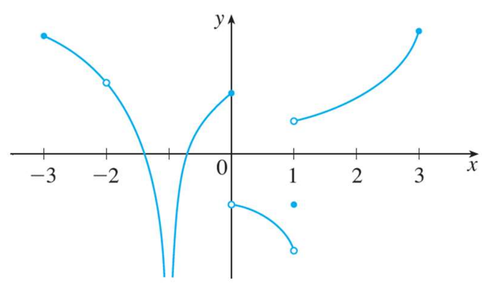
$x = -4$

$f(-4)$ is undefined

$x = -2$, $x = 2$, $x = 4$

$$\left.\begin{array}{l}\lim_{x \to -2} f(x) \\ \lim_{x \to 2} f(x) \\ \lim_{x \to 4} f(x)\end{array}\right\} \text{DNE}$$

- From the graph of g, state the intervals on which g is continuous.



$x = (-3, -2)$

$= (-2, -1)$

$= (-1, 0)$

$= (0, 1)$

$= (1, 3)$

- Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

(a) $g(t) = \frac{t^2+5t}{2t+1}, \quad a = 2$

$$g(2) = \frac{2^2+5.2}{2.2+1} = \frac{14}{5}$$

$$\lim_{t \to 2} \frac{t^2+5t}{2t+1} = \frac{14}{5}$$

$$\therefore \lim_{t \to 2} g(t) = g(2)$$

$\therefore$ $g(t)$ is continuous at $t = 2$

(b) $f(x) = 3x^4 - 5x + \sqrt[3]{x^2+4}, \quad a = 2$

$$f(2) = 3(2)^4 - 5 \cdot 2 + \sqrt[3]{2^2+4} = 40$$

$$\lim_{x \to 2} 3x^4 - 5x + \sqrt[3]{x^2+4} = 40$$

$$\therefore f(2) = \lim_{x \to 2} f(x)$$

$\therefore$ $f(x)$ is continuous at $x = 2$

- Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

$g(x) = \frac{x-1}{3x+6}$, $(-\infty, -2)$

$3x + 6 = 0$

$x = -2$

for $a < -2$

$\lim_{x \to a} \frac{x-1}{3x+6} = \frac{a-1}{3a+6}$

$g(a) = \frac{a-1}{3a+6}$

$\therefore\ g(a) = \lim_{x \to a} g(x)$

$\therefore\ g(x)$ is continuous on $(-\infty, -2)$

- Explain why the function is discontinuous at the given number a.

(a) $f(x) = \begin{cases} \frac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$ $\quad a = 1$

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x\cancel{(x-1)}}{\cancel{(x-1)}(x+1)}$$

$$= \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2}$$

$$\lim_{x \to 1} f(x) \neq f(1)$$

therefore $f(x)$ is discontinuous at $x = 1$

(b) $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$ $\quad a = 0$

$$\lim_{x \to 0^+} 1 - x^2 = 1$$

$$\lim_{x \to 0^-} \cos x = 1$$

$$\lim_{x \to 0} f(x) = 1$$

$$f(0) \neq \lim_{x \to 0} f(x)$$

therefore $f(x)$ is discontinuous at $x = 0$

- State the interval of continuity for the function.

(a) $Q(x) = \frac{\sqrt[3]{x-2}}{x^3-2}$

$D_{\sqrt[3]{x-2}} = \mathbb{R}$

لأن الجذر فردي

$x^3 - 2 = 0$

أصفار المقام

$x^3 = 2$

$x = \sqrt[3]{2}$

$D_Q = \mathbb{R} / \{\sqrt[3]{2}\}$

Q is continuous on $(-\infty, \sqrt[3]{2}) \cup (\sqrt[3]{2}, \infty)$

(b) $R(t) = \frac{e^{\sin t}}{2+\cos \pi t}$

$D_{e^{\sin t}} = \mathbb{R}$

exponential

$2 + \cos \pi t = 0$

$\cos \pi t = -2$

$-1 \leq \cos \theta \leq 1$

Has no solution

$D_R = \mathbb{R}$

R is continuous "everywhere" $\Rightarrow (-\infty, \infty)$

(c) $B(x) = \frac{\tan x}{\sqrt{4-x^2}}$

$D_{\tan x} = \mathbb{R} / \{ \frac{\pi}{2} + k\pi \ , \ k \in \mathbb{Z} \}$

$4 - x^2 > 0$

$-x^2 > -4$

$x^2 < 4$

$-2 < x < 2$

$D_{\sqrt{4-x^2}} = (-2, 2)$

$D_B = (-2, 2) \cap \mathbb{R} / \{ \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \}$

$D_B = (-2, 2)$

B is continuous on $(-2, 2)$

- **Use continuity to evaluate the limit**

$$\lim_{x\to 4} 3^{\sqrt{x^2-2x-4}}$$

$f(x) = 3^{\sqrt{x^2-2x-4}}$

is continuous on its domain

$x^2 - 2x - 4 \geq 0$

غيرقابلة للتحليل

$x^2 - 2x \geq 4$

completing the square

إكمال المربع

$x^2 - 2x + \left(\frac{-2}{2}\right)^2 \geq 4 + \left(\frac{-2}{2}\right)^2$

$x^2 - 2x + 1 \geq 5$

$(x-1)^2 \geq 5$

$x - 1 \geq \sqrt{5}$ or $x - 1 \leq -\sqrt{5}$

$x \geq 1 + \sqrt{5}$ $\qquad$ $x \leq 1 - \sqrt{5}$

$D_f = (-\infty, 1-\sqrt{5}) \cup (1+\sqrt{5}, \infty)$

$\therefore$ $x = 4$ is in the domain

$\therefore \lim_{x\to 4} f(x) = f(4) = 3^{\sqrt{4^2 - 2\cdot 4 - 4}}$

$= 3^2 = 9$

- **Show that f is continuous on $(-\infty, \infty)$.**

$$f(x) = \begin{cases} \sin x & \text{if} \quad x < \pi/4 \\ \cos x & \text{if} \quad x \geq \pi/4 \end{cases}$$

$f(x) = \sin x$ is continuos on $(-\infty, \frac{\pi}{4})$

$f(x) = \cos x$ is continuous on $(\frac{\pi}{4}, \infty)$

$$\lim_{x \to \pi/4^-} f(x) = \lim_{x \to \frac{\pi}{4}^-} \sin x = \frac{1}{\sqrt{2}}$$

$$\lim_{x \to \pi/4^+} f(x) = \lim_{x \to \frac{\pi}{4}^+} \cos x = \frac{1}{\sqrt{2}}$$

$$\lim_{x \to \pi/4} f(x) = f\left(\frac{\pi}{4}\right)$$

therefore $f(x)$ is continuous at $\frac{\pi}{4}$

therefore $f(x)$ is continuous on $(-\infty, \infty)$

- Find the numbers at which f is discontinuous.

$$f(x)=\begin{cases}2^x & \text{if} \quad x\le 1\\ 3-x & \text{if} \quad 1<x\le 4\\ \sqrt{x} & \text{if} \quad x>4\end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2^x = 2$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3-x = 2$$

$$\lim_{x\to 1} f(x) = f(1) = 2$$

$\therefore$ f is continuous at $x=1$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} 3-x = -1$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x} = 2$$

$\lim_{x\to 4} f(x)$ DNE

f is discontinuous at $x=4$

- Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if} \quad x < 2 \\ ax^2 - bx + 3 & \text{if} \quad 2 \le x < 3 \\ 2x - a + b & \text{if} \quad x \ge 3 \end{cases}$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-4}{x-2}$$

$$= \lim_{x\to 2^-} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = 4$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} ax^2 - bx + 3$$

for f to be continuous at $x = 2$

$$\lim_{x\to 2^+} ax^2 - bx + 3 = \lim_{x\to 2^-} f(x) = 4$$

$$4a - 2b + 3 = 4$$

$$4a = 1 + 2b \Rightarrow a = \frac{1+2b}{4} \rightarrow 1$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} ax^2 - bx + 3$$

$$= 9a - 3b + 3$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 2x - a + b = 6 - a + b$$

for f to be continuous at $x=3$

$$9a-3b+3=6-a+b$$

$$9a+a=3+4b$$

$$10a=3+4b$$

from equation ① $a=\dfrac{1+2b}{4}$

$$10\left(\frac{1+2b}{4}\right)=3+4b$$

$$\frac{10+20b}{4}=3+4b$$

$$10+20b=12+16b$$

$$4b=2 \Rightarrow b=\frac{2}{4}=\frac{1}{2}$$

$$a=\frac{1+2\left(\frac{1}{2}\right)}{4}=\frac{2}{4}=\frac{1}{2}$$

- **Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x \to 2}[3f(x) + f(x)g(x)] = 36$. Find $f(2)$.**

$$\lim_{x \to 2}[3f(x) + f(x)g(x)]$$

$$= 3\lim_{x \to 2} f(x) + \lim_{x \to 2} f(x) \cdot \lim_{x \to 2} g(x)$$

$\because$ f and g are continuous

$\therefore$ $\lim_{x \to 2} g(x) = g(2)$

and $\lim_{x \to 2} f(x) = f(2)$

therefore

$$3\lim_{x \to 2} f(x) + \lim_{x \to 2} f(x) \cdot \lim_{x \to 2} g(x)$$

$$= 3f(2) + f(2) \cdot g(2) = 36$$

$$\therefore \quad 3f(2) + 6f(2) = 36$$

$$9f(2) = 36$$

$$f(2) = \frac{36}{9} = 4$$

- Let $f(x) = 1/x$ and $g(x) = 1/x^2$.

(a) Find $(f \circ g)(x)$.

$$f(g(x)) = \frac{1}{\left(\frac{1}{x^2}\right)} = x^2$$

(b) Is $f \circ g$ continuous everywhere? Explain.

g is continuous on $\mathbb{R} / \{0\}$

f is continuous on $g \neq 0$

and since $g(x)$ can never equal 0

therefore $f \circ g$ is continuous on $\mathbb{R} / \{0\}$

$(-\infty, 0) \cup (0, \infty)$

- Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

$\ln x = x - \sqrt{x}, \qquad (2, 3)$

$\ln x - x + \sqrt{x} = 0$

$f(x) = \ln x - x + \sqrt{x}$ is continuous on $[2, 3]$

$f(2) = \ln 2 - 2 + \sqrt{2} > 0$

$f(3) = \ln 3 - 3 + \sqrt{3} < 0$

since $f(3) < 0 < f(2)$

therefore there exists a value c

such that $f(c) = 0$

- Prove that the equation has at least one real root.

$$x^4 + x = 3$$

$f(x) = x^4 + x - 3 = 0$ is continuos everywhere

$f(1) = 1 + 1 - 3 < 0$

$f(2) = 2^4 + 2 - 3 > 0$

$f(1) < 0 < f(2) \Rightarrow f(c) = 0 \Rightarrow x = c$ is a root