Section 2.5 – Continuity

- Continuity:

A function f is **continuous at a number a** if

$$\lim_{x\to a} f(x) = f(a)$$

① $f(a)$ defined

② $\lim_{x\to a} f(x)$ exists

③ $\lim_{x\to a} f(x) = f(a)$

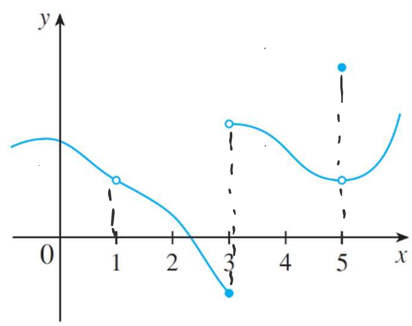
} شرط واحد يتضمن 3 شروط

discontinuous at:

$x = 1 \Rightarrow f(1)$ undefined

$x = 3 \Rightarrow \lim_{x\to 3} f(x)$ DNE

$x = 5 \Rightarrow \lim_{x\to 5} f(x) \neq f(5)$



Example 1

Where are each of the following functions discontinuous?

(a) $f(x) = \frac{x^2-x-2}{x-2}$

(b) $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Solution

(a) $f(2)$ not defined

$\therefore f(x)$ is discontinuous at $x = 2$

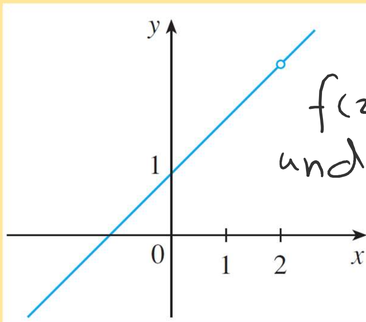
(b) $\lim_{x\to 0} \frac{1}{x^2}$ DNE

$\therefore f(x)$ is discontinuous at $x = 0$

لاحظ

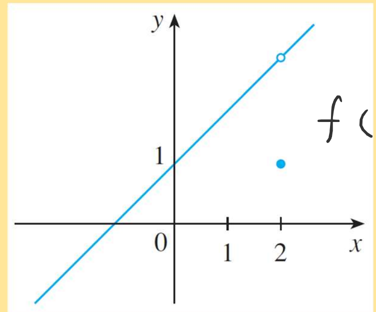
يوجد 3 أنواع لل discontinuity

1. Removable Discontinuity



$f(2)$ undefined

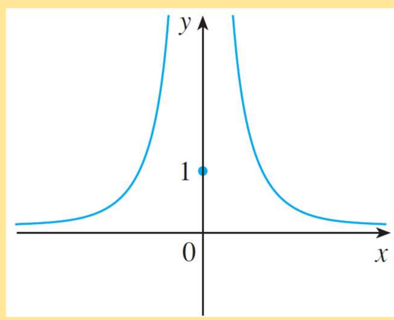
$$f(x) = \frac{x^2-x-2}{x-2}$$



$f(2) \neq \lim_{x \to 2} f(x)$

$$f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if } \quad x \neq 2 \\ 1 & \text{if } \quad x = 2 \end{cases}$$

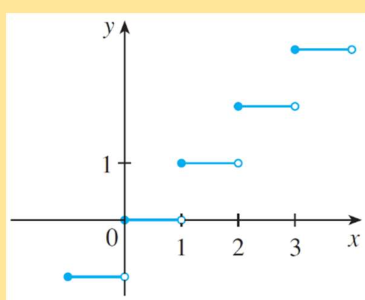
2. Infinite Discontinuity



$\lim_{x \to 0} f(x) = \infty$

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } \quad x \neq 0 \\ 1 & \text{if } \quad x = 0 \end{cases}$$

3. Jump Discontinuity



$\lim_{x \to a} f(x)$ DNE

$$f(x) = [\![x]\!]$$

- One-side continuity: اتصال من ناحية واحدة

A function f is **continuous from the right at a number a** if

$$\lim_{x\to a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x\to a^-} f(x) = f(a)$$

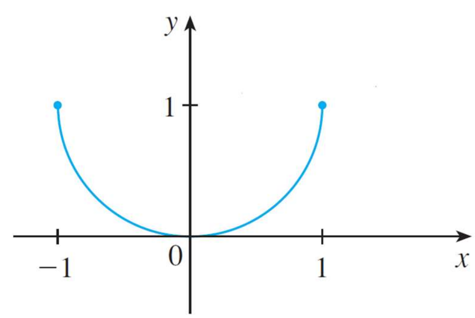
- Continuity on an interval:

A function f is **continuous on an interval (a, b)** if it is continuous at every number in the interval (on every point $x \in (a, b)$).

الدالة f متصلة في الفترة (a,b) إذا كانت متصلة عند كل القيم بين $x=a$ و $x=b$

$f(x)$ continuous on the interval $[-1,1]$

فترة مغلقة لأنها متصلة عند $x=-1$ من ناحية اليمين وعند $x=1$ من ناحية اليسار



Example 2

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1,1]$.

Solution

for $-1 \le a \le 1$

$$\lim_{x\to a} 1 - \sqrt{1-x^2} = 1 - \lim_{x\to a} \sqrt{1-x^2}$$

$$= 1 - \sqrt{\lim_{x\to a}(1-x^2)} = 1 - \sqrt{1-a^2}$$

also $f(a) = 1 - \sqrt{1-a^2}$

$\therefore f(a) = \lim_{x\to a} f(x) \Rightarrow f(x)$ is continuous for $x = [-1,1]$

- Theorem:

If f and g are continuous at a, and c is constant, then the following functions are also continuous at a:

1. $f+g$
2. $f-g$
3. cf
4. fg
5. $\frac{f}{g}$ if $g(a) \neq 0$

- Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

الحدوديات متصلة عند كل الأعداد الحقيقية

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

الدوال الكسرية متصلة عند كل قيم x المعرفة عندها

Example 3

Find $\lim_{x \to -2} \frac{x^3+2x^2-1}{5-3x}$

Solution

$f(x) = \frac{x^3+2x^2-1}{5-3x}$ is rational function

$\therefore f(x)$ is continuous on its domain

$\mathbb{R} / \{5/3\}$ → صفر المقام

$$\lim_{x \to -2} f(x) = f(-2) = \frac{(-2)^3+2(-2)^2-1}{5-3(-2)}$$

$$= -\frac{1}{11}$$

- Theorem:

The following types of functions are continuous at every number in their domains:

الدوال التالية متصلة عند كل القيم في مجالها:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

Example 4

Where is the function $f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

Solution

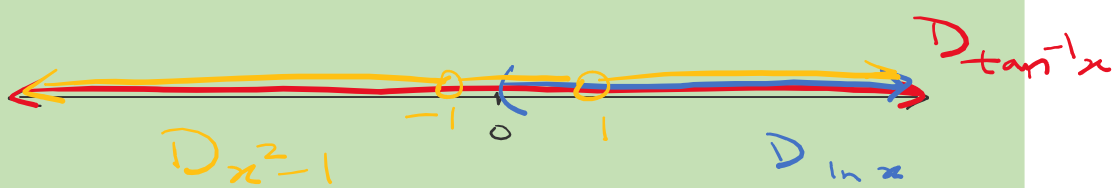
$D_{\ln x} = (0, \infty)$

$D_{\tan^{-1} x} = (-\infty, \infty)$

$D_{x^2-1} = \mathbb{R} / \{\pm 1\}$ أصفار المقام

$D_f = D_{\ln x} \cap D_{\tan^{-1} x} \cap D_{x^2-1}$

$= (0, 1) \cup (1, \infty)$



$f(x)$ is continuous on its domain

$(0,1) \cup (1, \infty) \equiv \mathbb{R}^+ / \{1\}$

- Theorem:

ليميت الدالة التركيبة

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

Example 5

Evaluate $\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$.

Solution

$$\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)$$

$$= \sin^{-1}\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\right)$$

$$= \sin^{-1}\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Example 6

Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$ (b) $F(x) = \ln(1 + \cos x)$

Solution

(a) $h(x) = f(g(x))$

$g(x) = x^2$ is continuous on $\mathbb{R}$

$f(x) = \sin x$ is continuous on $\mathbb{R}$

$\therefore$ $h(x)$ is continuous on $\mathbb{R}$

(b) $H(x) = 1 + \cos x$ is continuous on $\mathbb{R}$

$G(x) = \ln(1 + \cos x)$

is continuous on $1 + \cos x > 0$

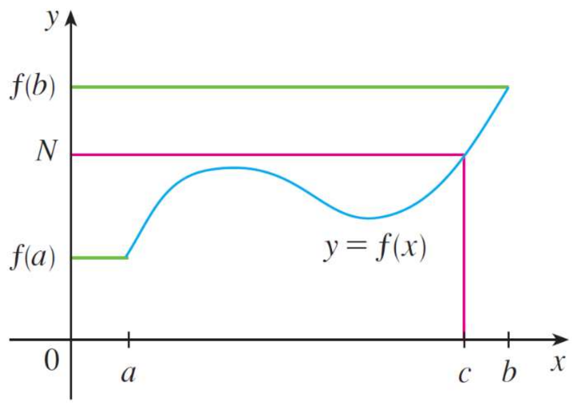
$\cos x > -1 \Rightarrow \cos x \neq -1$

$x \neq \pi + 2k\pi$, $k \in \mathbb{Z}$

- The Intermediate Value Theorem:

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$

$$f(a) < N < f(b) \quad \rightarrow \quad \exists c \in (a, b) \text{ such that } f(c) = N$$



Example 7

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

Solution

$f(1) = 4(1)^3 - 6(1)^2 + 3 \cdot 1 - 2 = -1$

$f(2) = 4(2)^3 - 6(2)^2 + 3 \cdot 2 - 2 = 12$

for $1 < c < 2$

$f(1) < f(c) < f(2)$

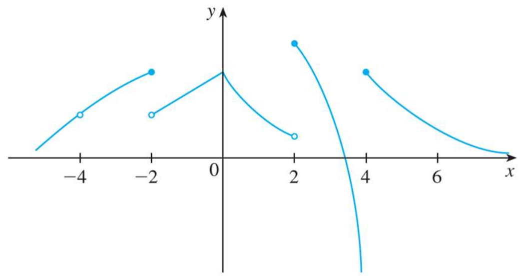
$-1 < f(c) < 12$

since $f(x)$ is continuous on $\mathbb{R}$

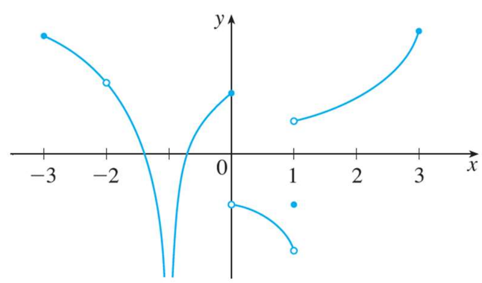
$\therefore f(c) = 0 \Rightarrow c$ is a root for $f(x)$

Problems

- From the graph of f, state the numbers at which f is discontinuous and explain why.



- From the graph of g, state the intervals on which g is continuous.



- Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

(a) $g(t) = \frac{t^2+5t}{2t+1}, \qquad a = 2$

(b) $f(x) = 3x^4 - 5x + \sqrt[3]{x^2+4}, \qquad a = 2$

- Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

$g(x) = \frac{x-1}{3x+6}$, $(-\infty, -2)$

- Explain why the function is discontinuous at the given number a.

(a) $f(x) = \begin{cases} \frac{x^2-x}{x^2-1} & \text{if} \quad x \neq 1 \\ 1 & \text{if} \quad x = 1 \end{cases}$ $\qquad a = 1$

(b) $f(x) = \begin{cases} \cos x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ 1 - x^2 & \text{if} \quad x > 0 \end{cases}$ $\qquad a = 0$

- State the interval of continuity for the function.

(a) $Q(x) = \frac{\sqrt[3]{x-2}}{x^3-2}$

(b) $R(t) = \frac{e^{\sin t}}{2+\cos \pi t}$

(c) $B(x) = \frac{\tan x}{\sqrt{4-x^2}}$

- Use continuity to evaluate the limit

$$\lim_{x\to 4} 3^{\sqrt{x^2-2x-4}}$$

- Show that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} \sin x & \text{if} \quad x < \pi/4 \\ \cos x & \text{if} \quad x \geq \pi/4 \end{cases}$$

- Find the numbers at which f is discontinuous.

$$f(x) = \begin{cases} 2^x & \text{if} \quad x \le 1 \\ 3 - x & \text{if} \quad 1 < x \le 4 \\ \sqrt{x} & \text{if} \quad x > 4 \end{cases}$$

- Find the values of a and b that make f continuous everywhere.

$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2} & \text{if} \quad x<2\\ ax^2-bx+3 & \text{if} \quad 2\le x<3\\ 2x-a+b & \text{if} \quad x\ge 3\end{cases}$$

- **Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x \to 2}[3f(x) + f(x)g(x)] = 36$. Find $f(2)$.**

- Let $f(x) = 1/x$ and $g(x) = 1/x^2$.

(a) Find $(f \circ g)(x)$.

(b) Is $f \circ g$ continuous everywhere? Explain.

- Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

$\ln x = x - \sqrt{x}, \qquad (2, 3)$

- Prove that the equation has at least one real root.

$$x^4 + x = 3$$