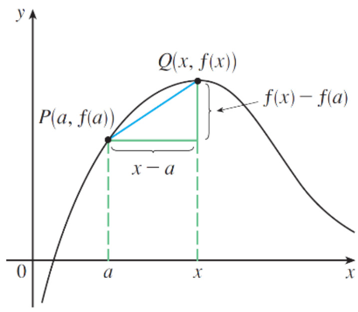
Section 2.7 – Derivatives and Rates of Change

- Slope of the secant line   خط يقطع المنحنى في نقطتين



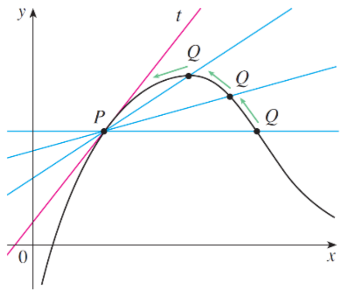
ميل الخط $m = \dfrac{f(x) - f(a)}{x - a} = \dfrac{y - y_0}{x - x_0}$

- Tangent line: خط يلمس المنحنى عند نقطة

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.



Example 1

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1,1)$.

Solution

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad a = 1 , \quad f(a) = 1$$

$$m = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}$$

$$= \lim_{x \to 1} x + 1 = 2$$

$$y - y_0 = m(x - x_0)$$

$$y - 1 = 2(x - 1) \Rightarrow y = 2x - 1$$

لاحظ

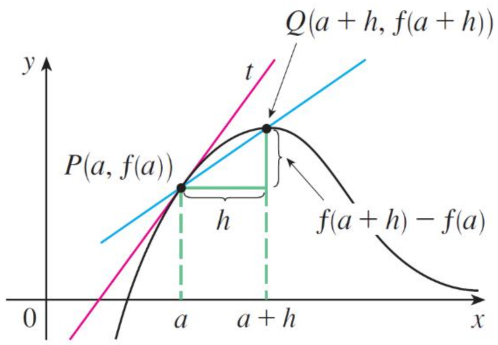
معادلة المماس
$$y - y_0 = m(x - x_0)$$

معادلة خط عمودي
$$y - y_0 = -\frac{1}{m}(x - x_0)$$

- h-definition

Take $h = x - a$ $\Rightarrow x = a + h$

$$m = \lim_{x\to a}\frac{f(x) - f(a)}{x - a} = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$



Example 2

Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3,1)$.

Solution

$$\lim_{h\to 0}\frac{f(a+h) - f(a)}{h}, \quad a = 3, \ f(a) = 1$$

$$f(a+h) = \frac{3}{3+h}$$

$$m = \lim_{h\to 0}\frac{\frac{3}{3+h} - 1}{h}$$

$$= \lim_{h\to 0}\frac{\frac{3}{3+h} - \frac{3+h}{3+h}}{h} = \lim_{h\to 0}\frac{\frac{\cancel{3} - \cancel{3} - h}{3+h}}{h}$$

$$= \lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}(3+h)} = -\frac{1}{3}$$

$$y - y_0 = m(x - x_0) \Rightarrow y - 1 = -\frac{1}{3}(x - 3)$$

- Derivative الاشتقاق:

The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

نفس معادلة
ميل الخط

is this limit exists.

Example 3

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(a+h)^2 - 8(a+h) + 9 - (a^2 - 8a + 9)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{8a} - 8h + \cancel{9} - \cancel{a^2} + \cancel{8a} - \cancel{9}}{h}$$

$$= \lim_{h\to 0} \frac{2ah + h^2 - 8h}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2a + h - 8)}{\cancel{h}}$$

$$= 2a - 8$$

Example 4

Find the equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

Solution

$y - y_0 = m(x - x_0)$, $m = f'(a)$

$m = 2a - 8$

$= 2 \cdot 3 - 8 = -2$

$y - (-6) = -2(x - 3)$

$y + 6 = -2x + 6$

$y = -2x$

لاحظ

$m = f'(a)$ لذن

$y - y_0 = f'(a)(x - x_0)$ معادلة المماس

Problems

- (a) Find the slope of the tangent line to the parabola $y = x - x^3$ at the point $(1, 0)$.

(i) using $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$

$$m = \lim_{x \to 1} \frac{x - x^3 - 0}{x - 1} = \lim_{x \to 1} \frac{-x(-1 + x^2)}{x - 1}$$

$$= \lim_{x \to 1} \frac{-x\cancel{(x-1)}(x+1)}{\cancel{x-1}}$$

$$= \lim_{x \to 1} -x(x+1) = -2$$

(ii) using $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$

$$m = \lim_{h \to 0} \frac{(1+h) - (1+h)^3 - 0}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{1} + h - (\cancel{1} + 3h + 3h^2 + h^3)}{h}$$

$$= \lim_{h \to 0} \frac{-2h + 3h^2 + h^3}{h}$$

$$= \lim_{h \to 0} -2 + 3h + h^2 = -2$$

(b) Find an equation of the tangent line in part (a).

$$y - y_0 = m(x - x_0)$$

$$y = -2(x - 1)$$

$$y = -2x + 2$$

- Find an equation of the tangent line to the curve at the given point.

(a) $y = x^3 - 3x + 1, \quad (2,3)$

$$m = \lim_{h \to 0} \frac{(2+h)^3 - 3(2+h) + 1 - 3}{h}$$

$$= \lim_{h \to 0} \frac{2^3 + 3 \cdot 2^2 \cdot h + 3 \cdot 2h^2 + h^3 - 6 - 3h - 2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{8} + 12h + 6h^2 + h^3 - \cancel{6} - 3h - \cancel{2}}{h}$$

$$= \lim_{h \to 0} \frac{9h + 6h^2 + h^3}{h}$$

$$= \lim_{h \to 0} 9 + 6h + h^2 = 9$$

(b) $y = \sqrt{x}, \ (2, \sqrt{2})$

$$m = \lim_{x \to 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2}$$

$$x - 2 = (\sqrt{x})^2 - (\sqrt{2})^2$$

$$= \lim_{x \to 2} \frac{\cancel{\sqrt{x} - \sqrt{2}}}{(\cancel{\sqrt{x} - \sqrt{2}})(\sqrt{x} + \sqrt{2})}$$

$$= \lim_{x \to 2} \frac{1}{\sqrt{x} + \sqrt{2}}$$

$$= \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$$

- (a) Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point where $x = a$.

$$m = \lim_{x \to a} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{\sqrt{a} - \sqrt{x}}{\sqrt{x}\,\sqrt{a}}}{x - a}$$

$$= \lim_{x \to a} \frac{\sqrt{a} - \sqrt{x}}{\sqrt{x}\,\sqrt{a}\,(x-a)}$$

$$= \lim_{x \to a} \frac{-\cancel{(\sqrt{x} - \sqrt{a})}}{\sqrt{x}\,\sqrt{a}\,\cancel{(\sqrt{x} - \sqrt{a})}(\sqrt{x} + \sqrt{a})}$$

$$= \lim_{x \to a} \frac{-1}{x\sqrt{a} + a\sqrt{x}} = \frac{-1}{2a\sqrt{a}}$$

(b) Find equations of the tangent lines at the points $(1, 1)$ and $\left(4, \frac{1}{2}\right)$.

$$y - y_0 = m(x - x_0)$$

point $(1,1)$, $m = -\frac{1}{2}$

$$y - 1 = \frac{-1}{2}(x - 1)$$

point $\left(4, \frac{1}{2}\right)$, $m = -\frac{1}{2 \cdot 4\sqrt{4}} = -\frac{1}{16}$

$$y - \frac{1}{2} = -\frac{1}{16}(x - 4)$$

- Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

$$y - y_0 = g'(a)(x - x_0)$$

$$y - (-3) = 4(x - 5)$$

$$y + 3 = 4x - 20$$

- If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.

$f(4) = 3$ from point $(4, 3)$

$$f'(4) = m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{2 - 3}{0 - 4} = \frac{-1}{-4} = \frac{1}{4}$$

- If $F(x) = 5x/(1+x^2)$, find $F'(2)$ and use it to find an equation of the tangent line to the curve $y = 5x/(1+x^2)$ at the point $(2, 2)$.

$$F'(x) = \lim_{h \to 0} \frac{\dfrac{5(2+h)}{1+(2+h)^2} - \dfrac{5(2)}{1+2^2}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{10+5h}{1+(2+h)^2} - 2}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{10+5h-2(1+(2+h)^2)}{1+(2+h)^2}}{h}$$

$$= \lim_{h \to 0} \frac{10+5h-2+2(2+h)^2}{h(1+(2+h)^2)}$$

$$= \lim_{h \to 0} \frac{10+5h-2-2(4+4h+h^2)}{h(1+(2+h)^2)}$$

$$= \lim_{h \to 0} \frac{\cancel{10}+5h-\cancel{2}-\cancel{8}-8h-h^2}{h(1+(2+h)^2)}$$

$$= \lim_{h \to 0} \frac{-3h-h^2}{h(1+(2+h)^2)}$$

$$= \lim_{h \to 0} \frac{-3-h}{1+(2+h)^2} = \frac{-3}{5}$$

- If $G(x) = 4x^2 - x^3$, find $G'(a)$ and use it to find equations of the tangent lines to the curve $y = 4x^2 - x^3$ at the points $(2,8)$ and $(3,9)$.

$$G'(a) = \lim_{h \to 0} \frac{4(a+h)^2 - (a+h)^3 - (4a^2 - a^3)}{h}$$

$$= \lim_{h \to 0} \frac{4(a^2 + 2ah + h^2) - (a^3 + 3 \cdot a^2 h + 3ah^2 + h^3) - (4a^2 - a^3)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4a^2} + 8ah + 4h^2 - \cancel{a^3} - 3a^2 h - 3ah^2 - h^3 - \cancel{4a^2} \cancel{+a^3}}{h}$$

$$= \lim_{h \to 0} \frac{8ah + 4h^2 - 3a^2 h - 3ah^2 - h^3}{h}$$

$$= \lim_{h \to 0} 8a + 4h - 3a^2 - 6ah - h^2$$

$$= 8a - 3a^2$$

$$y - y_0 = G'(a)(x - x_0)$$

at point $(2,8) \Rightarrow G'(2) = 8(2) - 3 \cdot 2^2 = 4$

$$y - 8 = 4(x - 2)$$

at point $(3,9) \Rightarrow G'(3) = 8(3) - 3 \cdot 3^2 = -3$

$$y - 9 = -3(x - 3)$$

- **Find $f'(a)$.**

(a) $f(t) = 2t^3 + t$

$$f'(a) = \lim_{h\to 0} \frac{2(a+h)^3 + (a+h) - (2a^3+a)}{h}$$

$$= \lim_{h\to 0} \frac{2(a^3+3a^2h+3ah^2+h^3)+a+h-2a^3-a}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2a^3}+6a^2h+6ah^2+2h^3+\cancel{a}+h-\cancel{2a^3}-\cancel{a}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(6a^2+6ah+2h^2+1)}{\cancel{h}}$$

$$= \lim_{h\to 0} 6a^2+6ah+2h^2+1$$

$$= 6a^2+1$$

(b) $f(x) = \frac{4}{\sqrt{1-x}}$

$$f'(a) = \lim_{h \to 0} \frac{\dfrac{4}{\sqrt{1-(a+h)}} - \dfrac{4}{\sqrt{1-a}}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{4\sqrt{1-a} - 4\sqrt{1-(a+h)}}{\sqrt{1-(a+h)}\,\sqrt{1-a}}}{h}$$

$$= \lim_{h \to 0} 4\, \frac{\sqrt{1-a} - \sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}} \cdot \frac{\sqrt{1-a} + \sqrt{1-a-h}}{\sqrt{1-a} + \sqrt{1-a-h}}$$

$$= \lim_{h \to 0} 4\, \frac{\cancel{1} - \cancel{a} - (\cancel{1} - \cancel{a} - h)}{h\sqrt{1-a-h}\,\sqrt{1-a}\,(\sqrt{1-a} + \sqrt{1-a-h})}$$

$$= \lim_{h \to 0} \frac{4\cancel{h}}{\cancel{h}\sqrt{1-a-h}\,\sqrt{1-a}\,(\sqrt{1-a} + \sqrt{1-a-h})}$$

$$= \frac{4}{\sqrt{1-a}\,\sqrt{1-a}\,(\sqrt{1-a} + \sqrt{1-a})}$$

$$= \frac{4}{(1-a)\,2\sqrt{1-a}} = \frac{2}{(1-a)\sqrt{1-a}}$$

- Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

(a) $\lim\limits_{h\to 0}\dfrac{e^{-2+h}-e^{-2}}{h}$

$$f'(x)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$f(a+h)=e^{-2+h}$ $\quad$ $f(a)=e^{-2}$

$f(x)=e^{x}$

$a=-2$

(b) $\lim\limits_{h\to 0}\dfrac{\cos(\pi+h)+1}{h}$

$$f'(x)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$f(a+h)=\cos(\pi+h)$ $\quad$ $f(a)=-1$

$f(x)=\cos\pi$

$a=\pi$