Section 2.8 – The Derivative as a Function

- Derivative of f at any value (x)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

معادلة عامة للاشتقاق (ليس عند نقطة محددة)

Example 1

If $f(x) = x^3 - x$, find a formula for $f'(x)$.

Solution

$$f'(x) = \lim_{h\to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x} - h - \cancel{x^3} + \cancel{x}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 1)}{\cancel{h}}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 1$$

$$f'(x) = 3x^2 - 1$$

لاحظ

يوجد أكثر من تعبير للمشتقة

كلهم نفس المعنى

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

Example 2

Find f' if $f(x) = \frac{1-x}{2+x}$.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1-(x+h)}{2+(x+h)} - \frac{1-x}{2+x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{(2+x+h)(2+x)}}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{2} - \cancel{x} - 2h - \cancel{x^2} - \cancel{xh}) - (\cancel{2} - \cancel{x} + h - \cancel{x^2} - \cancel{xh})}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{-3\cancel{h}}{\cancel{h}(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{-3}{(2+x+h)(2+x)}$$

$$= \frac{-3}{(2+x)^2}$$

- Differentiability القابلية للاشتقاق

A function f is **differentiable at a** if $f'(a)$ exists.

It is **differentiable on an open interval** (a, b) if it is differentiable at every number in the interval.

Example 3

Is the function $f(x) = |x|$ differentiable at $x = 0$?

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(0) = \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

$$\therefore \lim_{h \to 0^-} = \frac{-h}{h} = -1$$

$$\lim_{h \to 0^+} = \frac{h}{h} = 1$$

$$\therefore \lim_{h \to 0} \text{ DNE}$$

$f(x) = |x|$ is not differentiable at $x = 0$

- Theorem:

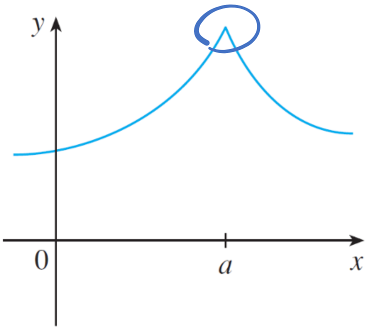
If f is differentiable at a, then f is continuous at a

لاحظ

Differentiability $\Longrightarrow$ continuity

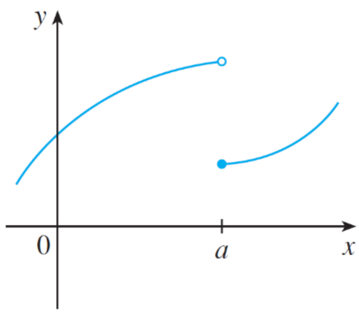
والعكس غير صحيح

٣ حالات تكون الدالة فيها غير قابلة للاشتقاق



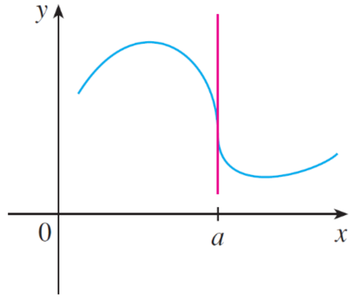
(a) A corner

زاوية حادة / ركن



(b) A discontinuity

عدم اتصال
(بكل حالاته)



(c) A vertical tangent

مماس عمودي

- Higher derivatives مشتقة المشتقة

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

Example 4

If $f(x) = x^3 - x$, find a formula for $f''(x)$.

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$$

$$\vdots$$

انظر مثال ١

$$f'(x) = 3x^2 - 1$$

$$f''(x) = \lim_{h\to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h}$$

$$= \lim_{h\to 0} \frac{3(x^2 + 2xh + h^2) \cancel{-1} - 3x^2 \cancel{+1}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{3x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(6x - 3h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 6x - 3h = 6x$$

Problems

- **Find the derivative of the function using the definition of derivative.**

(a) $f(x) = 4 + 8x - 5x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{4 + 8(x+h) - 5(x+h)^2 - [4 + 8x - 5x^2]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4} + \cancel{8x} + 8h - 5(x^2 + 2xh + h^2) - \cancel{4} - \cancel{8x} + 5x^2}{h}$$

$$= \lim_{h \to 0} \frac{8h - \cancel{5x^2} - 10xh - 5h^2 + \cancel{5x^2}}{h}$$

$$= \lim_{h \to 0} 8 - 10x - 5h = 8 - 10x$$

(b) $f(x) = x + \sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{x + h + \sqrt{x+h} - [x + \sqrt{x}]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h + \sqrt{x+h} - \cancel{x} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} + \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

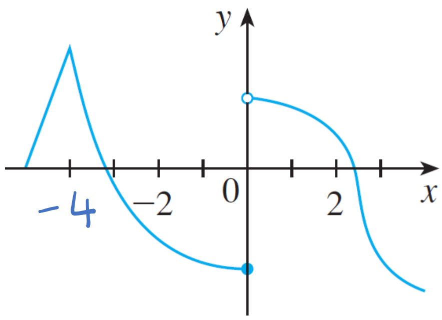
$$= \lim_{h \to 0} 1 + \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= 1 + \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= 1 + \frac{1}{2\sqrt{x}}$$

- The graph of f is given. State, with reasons, the numbers at which f is *not* differentiable.

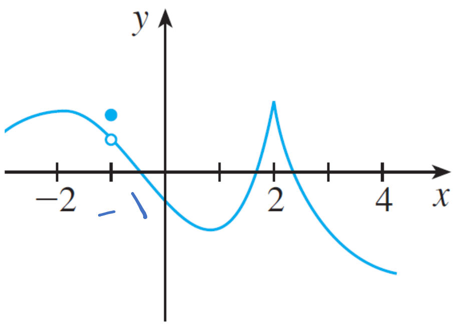
(a)



$x = -4$ corner

$x = 0$ discontinuity

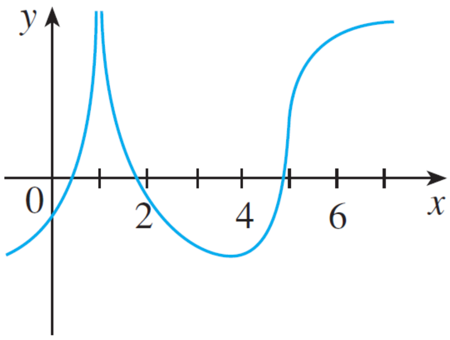
(b)



$x = -1$ discontinuity

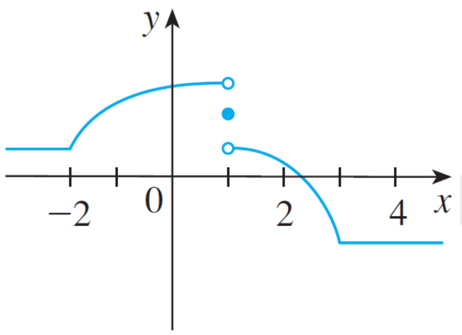
$x = 2$ corner

(c)



$x = 1$ discontinuity (VA)

(d)



$x = -2$ corner

$x = 1$ discontinuity

$x = 3$ corner

- (a) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.

$$g'(0) = \lim_{h \to 0} \frac{(0+h)^{2/3} - 0^{2/3}}{h}$$

$$= \lim_{h \to 0} \frac{h^{2/3}}{h} = \lim_{h \to 0} h^{2/3 - 1}$$

$$= \lim_{h \to 0} h^{2/3 - 1} = \lim_{h \to 0} h^{-1/3}$$

$$= \lim_{h \to 0} \frac{1}{h^{1/3}} \quad \text{DNE}$$

b) If $a \neq 0$, find $g'(a)$.

$$g'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{x^{2/3} - a^{2/3}}{x - a} = \lim_{x \to a} \frac{(x^{1/3})^2 - (a^{1/3})^2}{(x^{1/3})^3 - (a^{1/3})^3}$$

$$= \lim_{x \to a} \frac{\cancel{(x^{1/3} - a^{1/3})}(x^{1/3} + a^{1/3})}{\cancel{(x^{1/3} - a^{1/3})}(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$

$$= \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3} a^{-1/3} = \frac{2}{3a^{1/3}}$$

(c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0, 0)$.

$g(0)$ is defined, $g(0) = 0^{2/3} = 0$

and $g'(0) = \infty \Rightarrow y = x^{2/3}$ has a VA at $(0,0)$

- Show that the function $f(x) = |x - 6|$ is not differentiable at 6.
Find a formula for f'.

$$f'(6) = \lim_{h \to 0} \frac{\overbrace{|6+h-6|}^{f(6+h)} - \overbrace{|6-6|}^{f(6)}}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h} \qquad |h| = \begin{cases} h & h \geq 0 \\ -h & h < 0 \end{cases}$$

$$\lim_{h \to 0^+} \frac{h}{h} = 1 \neq \lim_{h \to 0^-} \frac{-h}{h} = -1$$

$$\therefore \lim_{h \to 0} \frac{|h|}{h} \quad \text{DNE}$$

and $f(x)$ is not differentiable at $x = 6$

- (a) Find $f'_-(4)$ and $f'_+(4)$ for the function

$$f(x) = \begin{cases} 0 & \text{if} \quad x \le 0 \\ 5 - x & \text{if} \quad 0 < x < 4 \\ \dfrac{1}{5 - x} & \text{if} \quad x \ge 4 \end{cases}$$

$$f'_-(4) = \lim_{h \to 0^-} \frac{f(4+h) - f(4)}{h}$$

$$= \lim_{h \to 0^-} \frac{5 - (4+h) - [5-4]}{h}$$

$$= \lim_{h \to 0^-} \frac{\cancel{1} - h - \cancel{1}}{h} = -1$$

$$f'_+(4) = \lim_{h \to 0^+} \frac{\dfrac{1}{5-(4+h)} - \dfrac{1}{5-4}}{h}$$

$$= \lim_{h \to 0^+} \frac{\dfrac{1}{1-h} - 1}{h}$$

$$= \lim_{h \to 0^+} \frac{\dfrac{1-(1-h)}{1-h}}{h}$$

$$= \lim_{h \to 0^+} \frac{\cancel{h}}{\cancel{h}(1-h)} = 1$$

(b) Where is f discontinuous?

$$\lim_{x \to 0^-} f(x) = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 5 - x = 5$$

$$\therefore \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

$$\therefore \lim_{x \to 0} f(x) \text{ DNE}$$

therefore $f(x)$ is discontinuous at $x = 0$

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} \frac{1}{5-x} = \infty$$

therefore $f(x)$ is discontinuous at $x = 5$ (VA)

(c) Where is f not differentiable?

f is not differentiable at

$x = 0$ and $x = 5$ discontinuity

$x = 4$ because $\lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$ DNE