Section 2.8 – The Derivative as a Function

- Derivative of f at any value (x)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

معادلة عامة للاشتقاق (ليس عند نقطة محددة a)

Example 1

If $f(x) = x^3 - x$, find a formula for $f'(x)$.

Solution

$$f'(x) = \lim_{h\to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$

$$= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 - 1)}{h}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 1$$

$$f'(x) = 3x^2 - 1$$

لاحظ

يوجد أكثر من تعبير للمشتقة

كلهم نفس المعنى

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

Example 2

Find f' if $f(x) = \frac{1-x}{2+x}$.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1-(x+h)}{2+(x+h)} - \frac{1-x}{2+x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{(2+x+h)(2+x)}}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{2} - \cancel{x} - 2h - \cancel{x^2} - \cancel{xh}) - (\cancel{2} - \cancel{x} + h - \cancel{x^2} - \cancel{xh})}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{-3\cancel{h}}{\cancel{h}(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{-3}{(2+x+h)(2+x)}$$

$$= \frac{-3}{(2+x)^2}$$

- Differentiability القابلية للاشتقاق

A function f is **differentiable at a** if $f'(a)$ exists.

It is **differentiable on an open interval** (a, b) if it is differentiable at every number in the interval.

Example 3

Is the function $f(x) = |x|$ differentiable at $x = 0$?

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(0) = \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

$$\because \lim_{h \to 0^-} = \frac{-h}{h} = -1$$

$$\lim_{h \to 0^+} = \frac{h}{h} = 1$$

$$\therefore \lim_{h \to 0} \text{ DNE}$$

$f(x) = |x|$ is not differentiable at $x = 0$

- Theorem:

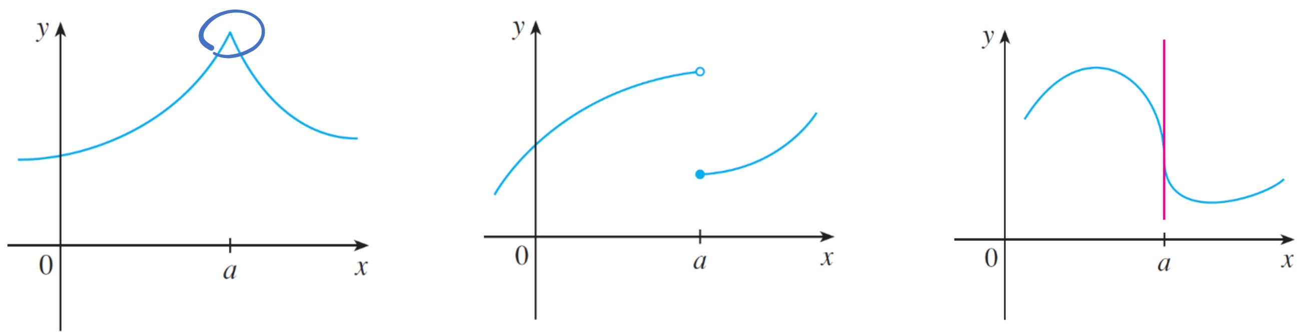
If f is differentiable at a, then f is continuous at a

لاحظ

Differentiability $\Longrightarrow$ continuity

والعكس غير صحيح

3 حالات تكون الدالة فيها غير قابلة للاشتقاق



(a) A corner — زاوية حادة / ركن

(b) A discontinuity — عدم اتصال (بكل حالاته)

(c) A vertical tangent — مماس عمودي

- Higher derivatives مشتقة المشتقة

$$f''(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}$$

Example 4

If $f(x) = x^3 - x$, find a formula for $f''(x)$.

Solution

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{[(x+h)^3-(x+h)]-[x^3-x]}{h}$$

$\vdots$

انظر مثال ١

$$f'(x) = 3x^2 - 1$$

$$f''(x) = \lim_{h\to 0}\frac{[3(x+h)^2-1]-[3x^2-1]}{h}$$

$$= \lim_{h\to 0}\frac{3(x^2+2xh+h^2)-1-3x^2+1}{h}$$

$$= \lim_{h\to 0}\frac{3x^2+6xh+3h^2-3x^2}{h}$$

$$= \lim_{h\to 0}\frac{h(6x-3h)}{h}$$

$$= \lim_{h\to 0} 6x-3h = 6x$$

Problems

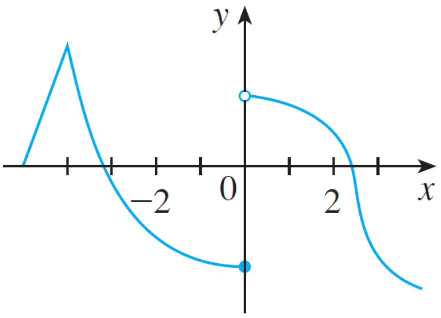
- Find the derivative of the function using the definition of derivative.

(a) $f(x) = 4 + 8x - 5x^2$

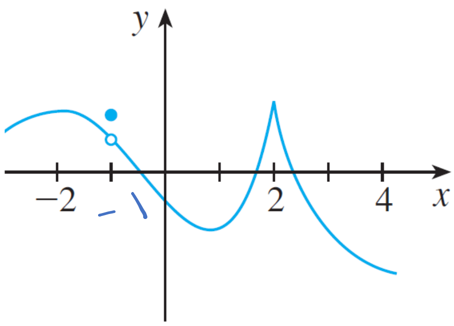
(b) $f(x) = x + \sqrt{x}$

- The graph of f is given. State, with reasons, the numbers at which f is *not* differentiable.

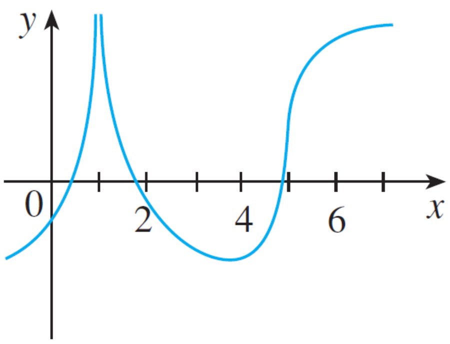
(a)



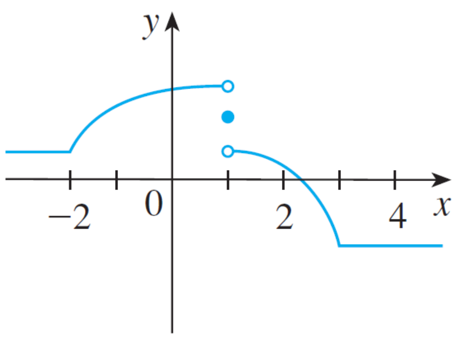
(b)



(c)



(d)



- (a) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.

b) If $a \neq 0$, find $g'(a)$.

(c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0, 0)$.

- Show that the function $f(x) = |x - 6|$ is not differentiable at 6.
Find a formula for f'.

- (a) Find $f'_{-}(4)$ and $f'_{+}(4)$ for the function

$$f(x) = \begin{cases} 0 & \text{if} \quad x \le 0 \\ 5 - x & \text{if} \quad 0 < x < 4 \\ \dfrac{1}{5 - x} & \text{if} \quad x \ge 4 \end{cases}$$

(b) Where is f discontinuous?

(c) Where is f not differentiable?