Section 3.1 – Derivatives of Polynomials and Exponential Functions

- Rules of differentiation

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example 1

Find the derivative for the following functions

(a) $f(x) = x^6$

(b) $y = x^{1000}$

(c) $f(x) = \frac{1}{x^2}$

(d) $\sqrt[3]{x^2}$

Solution

(a) $f'(x) = 6x^5$

(b) $\frac{dy}{dx} = 1000x^{999}$

(c) $f(x) = \frac{1}{x^2} = x^{-2}$

$f'(x) = -2x^{-3} = \frac{-2}{x^3}$

(d) $\sqrt[3]{x^2} = x^{2/3}$

$f' = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$

تذكر

ميل خط مستقيم عند نقطة هو نفسه قيمة الاشتقاق عند النقطة

$m = f'(a)$

Example 2

Find equations of the tangent line and normal line to the curve $y = x\sqrt{x}$ at the point $(1,1)$.

Solution

$y = x \cdot x^{1/2} = x^{3/2}$

$y' = \frac{3}{2} x^{1/2}$

$m = y'(1) = \frac{3}{2}$

$y - y_1 = m(x - x_1)$

$y - 1 = \frac{3}{2}(x - 1)$ tangent

$y - 1 = -\frac{2}{3}(x - 1)$ normal

- في حالة الضرب في ثابت:

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Example 3

Find the derivative of

(a) $y = 3x^4$

(b) $f(x) = -x$

Solution

(a) $y' = 3 \cdot 4x^3 = 12x^3$

(b) $\frac{d}{dx}f(x) = -1$

- في حالة الجمع والطرح:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Example 4

Let $f(x) = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5$

Solution

$f'(x) = 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$

Example 5

Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Solution

$y' = 4x^3 - 12x$

$4x^3 - 12x = 0$ horizontal line $\Rightarrow m = 0$

$4x(x^2 - 3) = 0$

$4x = 0$ or $x^2 - 3 = 0$

$x = 0$ $\quad$ $x = \pm\sqrt{3}$

The curve has horizontal tangent lines at

$x = 0$, $x = \sqrt{3}$, and $x = -\sqrt{3}$

The corresponding points are

$(0, 4)$

$(\sqrt{3}, -5)$

$(-\sqrt{3}, -5)$

- Exponential function rule

$$\frac{d}{dx}a^x = a^x \ln a$$

Example 6

Compute the derivative of the function $f(x) = a^x$ using the definition of the derivative.

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \qquad a^{x+h} = a^x \cdot a^h$$

$$= \lim_{h\to 0} \frac{a^x(a^h - 1)}{h}$$

$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

$$= a^x \ln a$$

$$\lim_{h\to 0} \frac{a^h - 1}{h} = \ln a \qquad \text{حفظ}$$

Example 7

Differentiate the function $j(x) = x^{24} + e^{24}$

Solution

$$j' = 24x^{23}$$

لاحظ

الأساس e حالة خاصة كالتالي:

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} e^{f(x)} = f'(x) \cdot e^{f(x)}$$

Problems

- Differentiate the function

(a) $f(x) = 186.5$

$f' = 0$

$\frac{d}{dx} c = 0$

(b) $f(x) = 5.2x + 2.3$

$f' = 5.2$

(c) $f(t) = 2t^3 - 3t^2 - 4t$

$f' = 6t^2 - 6t - 4$

(d) $g(x) = x^2(1 - 2x)$

$g(x) = x^2 - 2x^3$

$g' = 2x - 6x$

(e) $F(r) = \frac{5}{r^3}$

$$F(r) = 5r^{-3}$$

$$F' = -15r^{-4} = -\frac{15}{r^4}$$

(f) $R(a) = (3a+1)^2$

$$R(a) = 9a^2 + 6a + 1$$

$$R' = 18a + 6$$

(g) $y = 3e^x + \frac{4}{\sqrt[3]{x}}$

$$y = 3e^x + 4x^{-1/3}$$

$$y' = 3e^x + 4\cdot\left(-\frac{1}{3}\right)x^{-4/3} = 3e^x - \frac{4}{3x^{4/3}}$$

(h) $y = \frac{x^2+4x+3}{\sqrt{x}}$

$$y = \frac{x^2}{\sqrt{x}} + \frac{4x}{\sqrt{x}} + \frac{3}{\sqrt{x}} = x^{3/2} + 4x^{1/2} + 3x^{-1/2}$$

$$y' = \frac{3}{2}x^{1/2} + \frac{4}{2}x^{-1/2} - \frac{3}{2}x^{-3/2} = \frac{3}{2}x^{1/2} + \frac{2}{x^{1/2}} - \frac{3}{2x^{3/2}}$$

- **Find an equation of the tangent line to the curve at the given point**

(a) $y = 2e^x + x, \quad (0, 2)$

$y' = 2e^x + 1$

$m = y'(0) = 2\cancel{e^0}^{1} + 1 = 3$

$y - y_1 = m(x - x_1)$

$y - 2 = 3(x - 0)$

$y - 2 = 3x$

(b) $y = \sqrt[4]{x} - x, \quad (1, 0)$

$y' = \frac{1}{4}x^{-3/4} - 1$

$m = y'(1) = 1/4 - 1 = -\frac{3}{4}$

$y - 0 = -\frac{3}{4}(x - 1)$

$y = -3/4\, x + 3/4$

- **For what value of x does the graph of $f(x) = e^x - 2x$ have a horizontal tangent?**

$f' = e^x - 2$

$e^x - 2 = 0$

horizontal $\Rightarrow m = 0$

$e^x = 2$

$\ln e^x = \ln 2$

$x = \ln 2$

$x \cancel{\ln e}^{1} = \ln 2$

- **Show that the curve $y = 2e^x + 3x + 5x^3$ has no tangent line with slope 2.**

$y' = 2e^x + 3 + 15x^2$

$2e^x + 3 + 15x^2 = 2$

$2e^x + 15x^2 = -1$

$x^2 > 0$

$e^x > 0$

Has no solution

$\therefore$ y has no tangent with slope 2

- Find an equation of the tangent line to the curve $y = x^4 + 1$ that is parallel to the line $32x - y = 15$.

$-y = -32x + 15 \Rightarrow y = 32x - 15$

$m_1 = m_2 = 32$ parallel

$y' = 4x^3$

$4x^3 = 32$

$x^3 = 8$

$x = 2$

$y = 2^4 + 1 = 17$

$\therefore$ y has tangent with slope 32 at $(2, 17)$

$y - y_1 = m(x - x_1)$

$y - 17 = 32(x - 2)$

- Find an equation of the normal line to the curve $y = \sqrt{x}$ that is parallel to the line $2x + y = 1$.

for the parallel line

$y = -2x + 1$

$m_1 = -2 = m_2$

therefore the normal line to the curve

$y = \sqrt{x}$ has slope $\frac{1}{2}$

$y' = \frac{1}{2} x^{-1/2} = \frac{1}{2}$

$\therefore \quad x = 1$

$y = \sqrt{1} = 1$

the equation of the line

$y - 1 = -2(x - 1)$

- **Evaluate** $\lim_{x \to 1} \frac{x^{1000}-1}{x-1}$.

for $f(x) = x^{1000}$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

therefore

$$\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1} = f'(1)$$

using differentiation power rule

$$f'(x) = 1000x^{999}$$

$$f'(1) = 1000$$

- Evaluate $\lim_{x\to 3} \frac{e^{(x-3)^2}-1}{x^2+x-12}$.

$$\lim_{x\to 3} \frac{e^{(x-3)^2}-1}{(x+4)(x-3)}$$

$$= \lim_{x\to 3}\left[\frac{e^{(x-3)^2}-1}{(x+4)(x-3)} \cdot \frac{x-3}{x-3}\right]$$

$$= \lim_{x\to 3}\left[\frac{e^{(x-3)^2}-1}{(x-3)^2} \cdot \frac{x-3}{x+4}\right]$$

$$= \lim_{x\to 3} \frac{e^{(x-3)^2}-1}{(x-3)^2} \cdot \lim_{x\to 3} \frac{x-3}{x+4}$$

$$= 1 \cdot 0 = 0$$