Section 3.2 – The Product and Quotient Rules

- The Product Rule قاعدة الضرب

$$\frac{d}{dx}[f(x)\cdot g(x)] = fg' + f'g$$

Example 1

Find the derivative of the function

(a) $f(x) = xe^x$ (b) $g(x) = x^2(1-2x)$

Solution

(a) $f' = xe^x + 1.e^x = xe^x + e^x$

(b) $g' = x^2(-2) + 2x(1-2x)$

$= -2x^2 + 2x - 4x^2$

$= -6x^2 + 2x$

Example 2

If $f(x) = \sqrt{x}g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

Solution

$f'(x) = \sqrt{x}\, g'(x) + \frac{1}{2}x^{-1/2}\, g(x)$

$f'(4) = \sqrt{4}\, g'(4) + \frac{1}{2\sqrt{4}}\, g(4)$

$= 2\cdot 3 + \frac{1}{2\cdot 2}\cdot 2$

$= 6 + \frac{1}{2} = \frac{13}{2}$

- The Quotient Rule قاعدة القسمة

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{gf' - fg'}{g^2}$$

Example 3

Let $y = \frac{x^2+x-2}{x^3+6}$, find y'.

Solution

$$y' = \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

$$= \frac{2x^4 + x^3 + 12x + 6 - 3x^4 - 3x^3 + 6x^2}{(x^3+6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3+6)^2}$$

Example 4

Find an equation of the tangent line to the curve $y = \frac{e^x}{(1+x^3)}$ at the point $\left(1, \frac{1}{2}e\right)$.

Solution

$$y' = \frac{(1+x^3)e^x - e^x(3x^2)}{(1+x^3)^2}$$

$$m = y'(1) = \frac{2e - 3e}{4} = -\frac{e}{4}$$

$$y - \frac{1}{2}e = -\frac{e}{4}(x-1)$$

الخلاصة

$\frac{d}{dx}(c) = 0$ مشتقة الثابت	$\frac{d}{dx}(x^n) = nx^{n-1}$ قاعدة الأس	
$\frac{d}{dx}(a^x) = a^x \ln a$ Exponential function	$\frac{d}{dx}(e^x) = e^x$ base e	$\frac{d}{dx}\left(e^{f(x)}\right) = f'(x) \cdot e^{f(x)}$ e أس دالة (أي شيء غير x)
$\frac{d}{dx}(cf) = c \cdot f'$ دالة ضرب ثابت		
$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$ قاعدة الجمع (polynomial)	$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$ قاعدة الطرح (polynomial)	
$\frac{d}{dx}[f(x) \cdot g(x)] = fg' + f'g$ ضرب دالتين	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'g - fg'}{g^2}$ قسمة دالتين	

Problems

- Differentiate

(a) $g(x) = (x + 2\sqrt{x})e^x$

$$g' = (x + 2\sqrt{x})\, e^x + \left(1 + \frac{1}{\sqrt{x}}\right) e^x$$

$$2\sqrt{x} = 2x^{1/2}$$

$$f' = 2 \cdot \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{\sqrt{x}}$$

(b) $y = \frac{e^x}{1+x}$

$$y' = \frac{(1+x)e^x - e^x}{(1+x)^2}$$

(c) $f(t) = \frac{2t}{4-t^2}$

$$f' = \frac{(4-t^2) \cdot 2 - 2t(-2t)}{(4-t^2)^2}$$

$$= \frac{8 - 2t^2 + 4t^2}{(4-t^2)^2} = \frac{8 + 2t^2}{(4-t^2)^2}$$

(d) $J(v) = (v^3 - 2v)(v^{-4} + v^{-2})$

$$J' = (v^3 - 2v)(-4v^{-5} - 2v^{-3}) + (3v^2 - 2)(v^{-4} + v^{-2})$$

$$= \cancel{-4v^{-2}} - 2 + 8v^{-4} + \cancel{4v^{-2}} + 3v^{-2} + 3 - 2v^{-4} - 2v^{-2}$$

$$= 6v^{-4} + v^{-2} + 1 = \frac{6}{v^4} + \frac{1}{v^2} + 1$$

(e) $F(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$

$(y^{-2} - 3y^{-4})$

$$F' = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(1 + 15y^2) + (-2y^{-3} + 12y^{-5})(y + 5y^3)$$

$$= \frac{1}{y^2} + 15 - \frac{3}{y^4} - \frac{45}{y^2} - 2y^{-2} - 10 + 12y^{-4} + 60y^{-2}$$

$$= \frac{9}{y^4} + \frac{14}{y^2} + 5$$

(f) $y = \frac{\sqrt{x}}{2+x}$

$$y' = \frac{(2+\sqrt{x})\,\frac{1}{2}x^{-1/2} - \sqrt{x}}{(2+x)^2}$$

$$= \frac{\frac{1}{\sqrt{x}} + \frac{1}{2} - \sqrt{x}}{(2+x)^2}$$

(g) $y = \frac{t^3+3t}{t^2-4t+3}$

$$y' = \frac{(t^2-4t+3)(3t^2+3) - (t^2+3t)(2t-4)}{(t^2-4t+3)^2}$$

$$= \frac{3t^4+3t^2-12t^3-\cancel{12t}+9t^2+9-2t^3+4t^2-6t^2+\cancel{12t}}{(t^2-4t+3)^2}$$

$$= \frac{3t^4-14t^3+10t^2+9}{(t^2-4t+3)^2}$$

(h) $y = \frac{v^3-2v\sqrt{v}}{v}$

$$y' = \frac{v(3v^2-3\sqrt{v}) - (v^3-2v\sqrt{v})}{v^2}$$

$$= \frac{3v^3-3v\sqrt{v}-v^3+2v\sqrt{v}}{v^2}$$

$$= \frac{2v^3-v\sqrt{v}}{v^2} = 2v - \frac{\sqrt{v}}{v} = 2v - \frac{1}{\sqrt{v}}$$

حل آخر

$$y = \frac{v^3}{v} - \frac{2v\sqrt{v}}{v} = v^2 - 2\sqrt{v}$$

$$y' = 2v - \frac{1}{\sqrt{v}}$$

(i) $f(x) = \frac{x^2 e^x}{x^2+e^x}$

$$f' = \frac{(x^2+e^x)(x^2e^x + 2xe^x) - x^2e^x(2x+e^x)}{(x^2+e^x)^2}$$

$$= \frac{x^4e^x + \cancel{2x^3e^x} + \cancel{x^2e^{2x}} + 2xe^{2x} - \cancel{2x^3e^x} - \cancel{x^2e^{2x}}}{(x^2+e^x)^2}$$

$$= \frac{x^4e^x + 2xe^{2x}}{(x^2+e^x)^2}$$

- **Find $f'(x)$ and $f''(x)$**

(a) $f(x) = \sqrt{x}e^x$

$$f'(x) = \sqrt{x}\, e^x + \frac{1}{2\sqrt{x}} \cdot e^x$$

$$= \sqrt{x}\, e^x + \frac{e^x}{2\sqrt{x}}$$

$$f''(x) = \left(\sqrt{x}\, e^x + \frac{e^x}{2\sqrt{x}}\right) + \frac{2\sqrt{x}\, e^x - \frac{e^x}{\sqrt{x}}}{(2\sqrt{x})^2}$$

(b) $f(x) = \frac{x}{x^2-1}$

$$f'(x) = \frac{(x^2-1)\cdot 1 - x(2x)}{(x^2-1)^2}$$

$$= \frac{x^2 - 1 - 2x^2}{(x^2-1)^2} = \frac{-x^2 - 1}{x^4 - 2x^2 - 1}$$

$$f''(x) = \frac{(x^4-2x^2-1)(-2x) - (-x^2-1)(4x^3-4x)}{(x^4-2x^2-1)^2}$$

$$= \frac{-8x^5 + 4x^3 + 2x + 4x^5 - \cancel{4x^3} + \cancel{4x^3} - 4x}{(x^4-2x^2-1)^2}$$

$$= \frac{-4x^5 + 4x^3 - 2x}{(x^4-2x^2-1)^2}$$

- Find equations of the tangent line and normal line to the given curve at the specified point.

$$y = 2xe^x, \quad (0,0)$$

$$y' = 2xe^x + 2e^x$$

$$m = y'(0) = 2$$

$$y - y_1 = m(x - x_1)$$

$y = 2x$ tangent line

$y = -\frac{1}{2}x$ normal line

- If $g(x) = xf(x)$, where $f(3) = 4$ and $f'(3) = -2$, find an equation of the tangent line to the graph of g at the point where $x = 3$.

$$g'(x) = x f'(x) + f(x)$$

$$m = g'(3) = 3 \cdot f'(3) + f(3)$$

$$= 3 \cdot -2 + 4 = -2$$

$$g(3) = 3 \cdot 4 = 12$$

$$\therefore \quad y - 12 = -2(x - 3)$$

$$y - 12 = -2x + 6$$

$$y = -2x + 18$$