Section 3.3 – Derivatives of Trigonometric Functions

- Derivative of Sine and Cosine

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

Example 1

Differentiate

(a) $y = x^2 \sin x$ (b) $f(\theta) = \frac{\sin\theta}{1+\cos\theta}$

Solution

(a) $y' = x^2 \cos x + 2x \sin x$

(b) $f' = \frac{(1+\cos\theta)\cos\theta - \sin\theta(-\sin\theta)}{(1+\cos\theta)^2}$

$= \frac{\cos\theta + (\cos^2\theta + \sin^2\theta)}{(1+\cos\theta)^2}$ $= 1$

$= \frac{\cancel{\cos\theta + 1}}{(1+\cos\theta)^{\cancel{2}}} = \frac{1}{1+\cos\theta}$

- Special limits of trigonometric functions

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x}{\sin x} = 1$$

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$$

Example 2

Find the limit

(a) $\lim_{x \to 0} \frac{\sin 7x}{4x}$ (b) $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\sin \theta}$

Solution

(a) $\lim_{x \to 0} \frac{\sin 7x}{4x} \cdot \frac{7}{7}$

$= \lim_{x \to 0} \frac{\sin 7x}{7x} \cdot \frac{7}{4}$

$= \frac{7}{4} \lim_{x \to 0} \frac{\sin(7x)}{(7x)} = \frac{7}{4} \cdot 1 = \frac{7}{4}$

(b) $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} \cdot \frac{\theta}{\sin \theta}$

$= \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} \cdot \lim_{\theta \to 0} \frac{\theta}{\sin \theta}$

$= 0 \cdot 1 = 0$

Example 3

Prove, using the definition of derivatives, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \left[\frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h} \right]$$

$$= \lim_{h\to 0} \left[\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right]$$

$$= \cos x \, \cancel{\lim_{h\to 0} \frac{\cos h - 1}{h}}^{\,0} - \sin x \, \cancel{\lim_{h\to 0} \frac{\sin h}{h}}^{\,1}$$

$$= -\sin x$$

تذكر

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

- Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Example 4

Differentiate $f(x) = \frac{\sec x}{1+\tan x}$.

Solution

$$f'(x) = \frac{(1+\tan x)\sec x \tan x - \sec x(\sec^2 x)}{(1+\tan)^2}$$

$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$

تذكر

$$\sin^2\theta \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Example 5

Find the limit $\lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$

Solution

$$\lim_{x\to 0} x \cdot \sin\left(\frac{1}{x}\right)$$

$$-1 \leq \sin\frac{1}{x} \leq 1$$

$$-x \leq x\sin\frac{1}{x} \leq x$$

$$\lim_{x\to 0} -x = \lim_{x\to 0} x = 0$$

$\therefore \lim_{x\to 0} x\sin\frac{1}{x} = 0$ by sandwich theorem

Example 6

Find the 27th derivative of $\cos x$.

Solution

$$f' = -\sin x$$

$$f'' = -\cos x$$

$$f''' = \sin x$$

$$f^{(4)} = \cos x$$

therefore $f^{(n)} = \cos x$ for n is a multiple of 4

$$f^{(24)} = \cos x$$

$$f^{(25)} = -\sin x$$

$$f^{(26)} = -\cos x \Rightarrow f^{(27)} = \sin x$$

Problems

- Differentiate

(a) $y = 2\sec x - \csc x$

$$y' = 2\sec x \tan x + \csc x \cot x$$

(b) $g(t) = 4\sec t + \tan t$

$$g'(t) = 4\sec t \tan t + \sec^2 t$$

(c) $y = e^u(\cos u + cu)$

$$y' = e^u(-\sin u + c) + e^u(\cos u + cu)$$

(d) $y = \frac{x}{2 - \tan x}$

$$y' = \frac{(2-\tan x)\cdot 1 - x(-\sec^2 t)}{(2-\tan x)^2}$$

$$= \frac{2 - \tan x + x\sec^2 t}{(2-\tan x)^2}$$

(e) $y = \sin\theta\cos\theta$

$$y' = \sin\theta \cdot -\sin\theta + \cos\theta\cos\theta$$

$$= -\sin^2\theta + \cos^2\theta$$

(f) $y = \frac{\cos x}{1-\sin x}$

$$y' = \frac{(1-\sin x)\cdot -\sin x - \cos x(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$$

$$= \frac{\cancel{-\sin x + 1}}{(1-\sin x)^{\cancel{2}}} = \frac{1}{1-\sin x}$$

(g) $f(t) = \frac{\sin t}{1+\tan t}$

$$f'(t) = \frac{(1+\tan t)\cos t - \sin t(\sec^2 t)}{(1+\tan t)^2}$$

$$= \frac{\cos t + \frac{\sin t}{\cancel{\cos t}}\cancel{\cos t} - \sin t\,\frac{1}{\cos^2 t}}{(1+\tan t)^2}$$

$$= \frac{\cos t + \sin t - \tan t\sec t}{(1+\tan t)^2}$$

- Prove that

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$= \frac{\sin x \cdot 0 - \cos x}{(\sin x)^2}$$

$$= \frac{-\cos x}{\sin x \ \sin x}$$

$$= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$= -\csc x \cdot \cot x$$

- **Find an equation of the tangent line to the curve $y = e^x \cos x$ at (0.1).**

$$y' = e^x \cdot -\sin x + e^x \cos x$$

$$m = y'(0) = e^0 \cdot \cancel{-\sin 0} + \cancel{e^0 \cos 0}$$

$$= 1$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 1(x - 0)$$

$$y - 1 = x \quad \Rightarrow \quad y = x + 1$$

- **If $H(\theta) = \theta \sin\theta$, find $H'(\theta)$ and $H''(\theta)$.**

$$H'(\theta) = \underbrace{\theta \cdot \cos\theta}_{} + \underbrace{1 \cdot \sin\theta}_{}$$

$$fg' + f'g$$

$$H''(\theta) = \theta \cdot -\sin\theta + 1 \cdot \cos\theta + \cos\theta$$

$$= -\theta \sin\theta + 2\cos\theta$$

- For what values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent?

$$f'(x) = 1 + 2\cos x$$

$$1 + 2\cos x = 0$$

$$\cos x = -\frac{1}{2}$$

Sin	All
tan	cos

$\cos x = \frac{1}{2}$ for $x = \frac{\pi}{3}$

$\therefore$ $\cos x = -\frac{1}{2}$ for $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ $\Rightarrow$ quadrant II

and $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ $\Rightarrow$ quadrant III

therefore $x = \frac{2\pi}{3} + 2k\pi$

or $x = \frac{4\pi}{3} + 2k\pi$, $k \in \mathbb{Z}$

- Find the limit

(a) $\lim\limits_{x\to 0} \dfrac{\sin x}{\sin \pi x}$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\pi x}{\sin \pi x} \cdot \frac{x}{\pi x}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\pi x}{\sin \pi x} \cdot \lim_{x\to 0} \frac{\cancel{x}}{\pi \cancel{x}}$$

$$= 1 \cdot 1 \cdot \frac{1}{\pi} = \frac{1}{\pi}$$

(b) $\lim\limits_{t\to 0} \dfrac{\tan 6t}{\sin 2t}$

$$= \lim_{t\to 0} \frac{\frac{\sin 6t}{\cos 6t}}{\sin 2t} = \lim_{t\to 0} \frac{\sin 6t}{\sin 2t \; \cos 6t}$$

$$= \lim_{t\to 0} \frac{\sin 6t}{6t} \cdot \frac{2t}{\sin 2t} \cdot \frac{1}{\cos 6t} \cdot \frac{6\cancel{t}}{2\cancel{t}}$$

$$= 1 \cdot 1 \cdot \frac{1}{\underbrace{\cancel{\cos 0}}_{1}} \cdot 3 = 3$$

(c) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{5x^3 - 4x}$

$$= \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{3x}{5x^3 - 4x}$$

$$= \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \lim_{x\to 0} \frac{3\cancel{x}}{\cancel{x}(5x^2 - 4)}$$

$$= 1 \cdot \frac{3}{-4} = -\frac{3}{4}$$

(d) $\lim\limits_{x\to\pi/4} \frac{1-\tan x}{\sin x-\cos x}$

$$= \lim_{x\to\pi/4} \frac{1-\frac{\sin x}{\cos x}}{\sin x-\cos x}\cdot\frac{\cos x}{\cos x}$$

$$= \lim_{x\to\pi/4} \frac{\cos x-\sin x}{(\sin x-\cos x)\cos x}$$

$$= \lim_{x\to\pi/4} \frac{-1}{\cos x} = \frac{-1}{1/\sqrt{2}} = -\sqrt{2}$$

(e) $\lim\limits_{x\to 1} \frac{\sin(x-1)}{x^2+x-2}$

$$= \lim_{x\to 1} \frac{\sin(x-1)}{(x+2)(x-1)}$$

$$= \lim_{x\to 1} \frac{\sin(x-1)}{x-1}\cdot\lim_{x\to 1}\frac{1}{x+2}$$

$$= 1\cdot\frac{1}{3} = \frac{1}{3}$$

- **Find the derivative $\frac{d^{99}}{dx^{99}}(\sin x)$**

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d^2}{dx^2}(\sin x) = -\sin x$$

$$\frac{d^3}{dx^3}(\sin x) = -\cos x$$

$$\frac{d^4}{dx^4}(\sin x) = \sin x$$

therefore $\frac{d^n}{dx^n}(\sin x) = \sin x$

for n is multiple of 4

$$\frac{d^{96}}{dx^{96}}(\sin x) = \sin x$$

$$\frac{d^{97}}{dx^{97}}(\sin x) = \cos x$$

$$\frac{d^{98}}{dx^{98}}(\sin x) = -\sin x$$

$$\frac{d^{99}}{dx^{99}}(\sin x) = -\cos x$$