Section 3.3 – Derivatives of Trigonometric Functions

- Derivative of Sine and Cosine

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

Example 1

Differentiate

(a) $y = x^2 \sin x$ (b) $f(\theta) = \frac{\sin\theta}{1+\cos\theta}$

Solution

(a) $y' = x^2 \cos x + 2x \sin x$

(b) $f' = \frac{(1+\cos\theta)\cos\theta - \sin\theta(-\sin\theta)}{(1+\cos\theta)^2}$

$= \frac{\cos\theta + (\cos^2\theta + \sin^2\theta)}{(1+\cos\theta)^2}$ $= 1$

$= \frac{\cancel{\cos\theta + 1}}{(1+\cos\theta)^{\cancel{2}}} = \frac{1}{1+\cos\theta}$

- Special limits of trigonometric functions

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{x}{\sin x}=1$$

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0$$

Example 2

Find the limit

(a) $\lim_{x\to 0}\frac{\sin 7x}{4x}$ (b) $\lim_{\theta\to 0}\frac{\cos\theta-1}{\sin\theta}$

Solution

(a) $\lim_{x\to 0}\frac{\sin 7x}{4x}\cdot\frac{7}{7}$

$=\lim_{x\to 0}\frac{\sin 7x}{7x}\cdot\frac{7}{4}$

$=\frac{7}{4}\lim_{x\to 0}\frac{\sin(7x)}{(7x)}=\frac{7}{4}\cdot 1=\frac{7}{4}$

(b) $\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}\cdot\frac{\theta}{\sin\theta}$

$=\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin\theta}$

$=0\cdot 1=0$

Example 3

Prove, using the definition of derivatives, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h} \right]$$

$$= \lim_{h \to 0} \left[\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right]$$

$$= \cos x \lim_{h \to 0} \frac{\cos h - 1}{h} \,(=0) - \sin x \lim_{h \to 0} \frac{\sin h}{h} \,(=1)$$

$$= -\sin x$$

تذكر

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

- Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Example 4

Differentiate $f(x) = \frac{\sec x}{1+\tan x}$.

Solution

$$f'(x) = \frac{(1+\tan x)\sec x \tan x - \sec x(\sec^2 x)}{(1+\tan)^2}$$

$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$

تذكر

$$\sin^2\theta \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Example 5

Find the limit $\lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$

Solution

$$\lim_{x\to 0} x \cdot \sin\left(\frac{1}{x}\right)$$

$$-1 \le \sin\frac{1}{x} \le 1$$

$$-x \le x\sin\frac{1}{x} \le x$$

$$\lim_{x\to 0} -x = \lim_{x\to 0} x = 0$$

$\therefore \lim_{x\to 0} x\sin\frac{1}{x} = 0$ by sandwich theorem

Example 6

Find the 27th derivative of $\cos x$.

Solution

$f' = -\sin x$

$f'' = -\cos x$

$f''' = \sin x$

$f^{(4)} = \cos x$

therefore $f^{(n)} = \cos x$ for n is a multiple of 4

$f^{(24)} = \cos x$

$f^{(25)} = -\sin x$

$f^{(26)} = -\cos x \Rightarrow f^{(27)} = \sin x$

Problems

- Differentiate

(a) $y = 2\sec x - \csc x$

(b) $g(t) = 4\sec t + \tan t$

(c) $y = e^{u}(\cos u + cu)$

(d) $y = \frac{x}{2-\tan x}$

(e) $y = \sin\theta\cos\theta$

(f) $y = \frac{\cos x}{1-\sin x}$

(g) $f(t) = \frac{\sin t}{1+\tan t}$

- Prove that

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

- Find an equation of the tangent line to the curve $y = e^x \cos x$ at (0.1).

- If $H(\theta) = \theta \sin \theta$, find $H'(\theta)$ and $H''(\theta)$.

- For what values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent?

- Find the limit

(a) $\lim_{x \to 0} \frac{\sin x}{\sin \pi x}$

(b) $\lim_{t \to 0} \frac{\tan 6t}{\sin 2t}$

(c) $\lim_{x \to 0} \frac{\sin 3x}{5x^3 - 4x}$

(d) $\lim_{x \to \pi/4} \frac{1 - \tan x}{\sin x - \cos x}$

(e) $\lim_{x \to 1} \frac{\sin(x-1)}{x^2 + x - 2}$

- Find the derivative $\frac{d^{99}}{dx^{99}}(\sin x)$