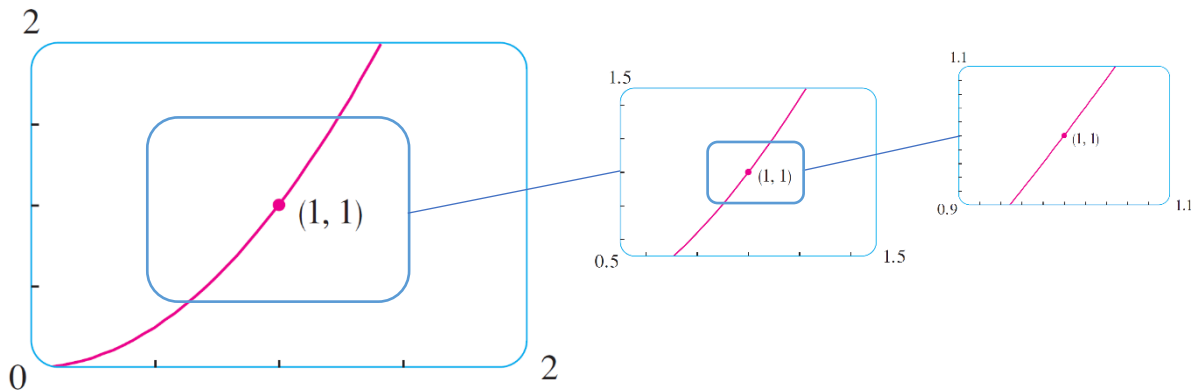


Section 3.10 – Linear Approximations and Differentials



- كلما اقتربنا من نقطة على منحنى الدالة ما يبدو شكل المنحنى كأنه خط مستقيم

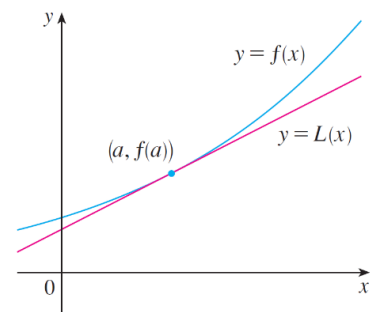
- معادلة المماس tangent عند أي نقطة على رسم دالة هي:

$$y - y_0 = m(x - x_0)$$

$$y - f(a) = f'(a)(x - a)$$

$$y = f(a) + f'(a)(x - a)$$

$$L(x) = f(a) + f'(a)(x - a)$$



Linearization of f at a

نستطيع استخدام نفس المعادلة للحصول على قيمة تقريبية لأي نقطة قريبة منها على رسم الدالة

Linear approximation or tangent line approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

Example 1

Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

Solution

$$f'(x) = \frac{1}{2}(x+3)^{-1/2}$$

$$f'(1) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$f(1) = \sqrt{4} = 2$$

$$\begin{aligned}\therefore L(x) &= 2 + \frac{1}{4}(x-1) \\ &= 2 + \frac{1}{4}x - \frac{1}{4} = \frac{x}{4} + \frac{7}{4}\end{aligned}$$

$$\therefore f(x) \approx \frac{x}{4} + \frac{7}{4} \quad \text{for } x \text{ near } 1$$

$$\sqrt{3.98} \approx \frac{0.98}{4} + \frac{7}{4} = \frac{7.98}{4} = 1.995$$

$$\sqrt{4.05} \approx \frac{1.05}{4} + \frac{7}{4} = \frac{8.05}{4} = 2.0125$$

Example 2

Estimate $\sqrt{99.8}$.

Solution

$$f(x) = \sqrt{x}, \quad a = 100, \quad f(a) = \sqrt{100} = 10$$

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$\sqrt{99.8} \approx 10 + \frac{1}{20}(99.8 - 100)$$

$$\approx 10 - \frac{0.2}{20}$$

$$\approx 10 - 0.01 \approx 9.99$$

Example 3

Approximate $(2.001)^5$.

Solution

$$f(x) = x^5, \quad a = 2, \quad f(a) = 2^5 = 32$$

$$f'(x) = 5x^4, \quad f'(2) = 5 \cdot 2^4 = 80$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$(2.001)^5 \approx 32 + 80(2.001 - 2)$$

$$\approx 32 + 0.08$$

$$\approx 32.08$$

Example 4

Use linear approximation to show that if $x \approx 1$, then $\sqrt{x} \approx \frac{1}{2}(x + 1)$

Solution

$$f(x) = \sqrt{x} \Rightarrow f(1) = 1$$

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(1) = \frac{1}{2}$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$\sqrt{x} \approx 1 + \frac{1}{2}(x-1)$$

$$\approx 1 + \frac{1}{2}x - \frac{1}{2}$$

$$\approx \frac{1}{2} + \frac{1}{2}x$$

$$\approx \frac{1}{2}(x+1)$$

- Differentials

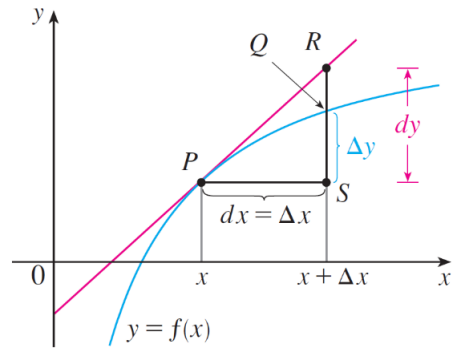
$$dy = f'(x) dx$$

مشتقة x →
مشتقة y ←

dy : قيمة تغير y التقريبية
نتيجة تغير x

$$\Delta y = f(x + \Delta x) - f(x)$$

Δy : قيمة تغير y الدقيقة
نتيجة تغير x



Example 5

Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes from 2 to 2.05.

Solution

$$f(2) = 2^3 + 2^2 - 2 \cdot 2 + 1 = 9$$

$$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1 = 9.7176$$

$$\Delta y = f(2.05) - f(2) = 0.7176$$

$$f'(x) = 3x^2 + 2x - 2$$

$$dy = f'(x) dx$$

$$= (3 \cdot 2^2 + 2 \cdot 2 - 2) \cdot 0.05$$

$$= 14 \times 0.05 = 0.7$$

$$x = 2$$

$$dx = \Delta x = 0.05$$

Problems

- Find the linearization $L(x)$ of $f(x) = \sin x$ at $a = \pi/6$.

$$f(a) = \sin \frac{\pi}{6} = \frac{1}{2}$$

$$f'(x) = \cos x \Rightarrow f'(a) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$L(x) = f(a) + f'(a)(x - a)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}\pi}{12} + \frac{1}{2}$$

- Find the differential of each function.

(a) $y = xe^{-4x}$

$$dy = f'(x) dx$$

$$f(x) = y = xe^{-4x}$$

$$f'(x) = -4xe^{-4x} + e^{-4x}$$

$$dy = (-4xe^{-4x} + e^{-4x}) dx$$

(b) $y = \frac{1+2u}{1+3u}$

$$dy = f'(u) du$$

$$f(u) = y = \frac{1+2u}{1+3u}$$

$$f'(u) = \frac{(1+3u)2 - (1+2u)3}{(1+3u)^2}$$

$$= \frac{2 + \cancel{6u} - 3 - \cancel{6u}}{(1+3u)^2} = \frac{-1}{(1+3u)^2}$$

$$dy = \frac{-1}{(1+3u)^2} du$$

- Use a linear approximation to estimate the given number.

(a) $1/4.002$

$$f(x) = \frac{1}{x}, \quad f'(x) = -\frac{1}{x^2}$$

$$a = 4, \quad f(a) = \frac{1}{4}, \quad f'(a) = -\frac{1}{16}$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$f(4.002) \approx \frac{1}{4} - \frac{1}{16}(4.002 - 4)$$

$$\approx \frac{1}{4} - \frac{1}{16} \cdot \frac{2}{1000}$$

$$\approx \frac{1}{4} - \frac{1}{8000} = \frac{1999}{8000}$$

(b) $\cos 29^\circ$

$$f(x) = \cos x, \quad f'(x) = -\sin x$$

$$a = 30^\circ, \quad f(a) = \cos(30^\circ) = \frac{\sqrt{3}}{2}, \quad f'(a) = -\sin(30^\circ) = -\frac{1}{2}$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$f(29^\circ) \approx \frac{\sqrt{3}}{2} - \frac{1}{2}(29^\circ - 30^\circ) \approx \frac{\sqrt{3}}{2} - \frac{1}{2}\left(-\frac{\pi}{180}\right)$$

$$\approx \frac{\sqrt{3}}{2} + \frac{\pi}{360}$$