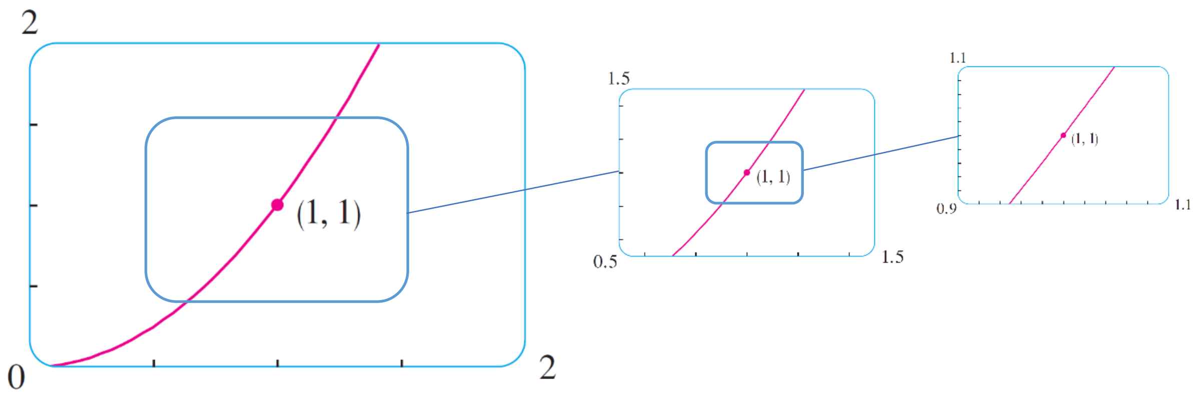
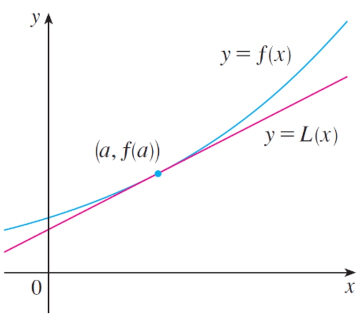
Section 3.10 – Linear Approximations and Differentials



- كلما اقتربنا من نقطة على منحنى الدالة يبدو شكل المنحنى كأنه خط مستقيم

- معادلة المماس tangent عند أي نقطة على رسم دالة هي:

$$y - y_0 = m(x - x_0)$$
$$y - f(a) = f'(a)(x - a)$$
$$y = f(a) + f'(a)(x - a)$$
$$L(x) = f(a) + f'(a)(x - a)$$



Linearization of $\underline{f}$ at $\underline{a}$

نستطيع استخدام نفس المعادلة للحصول على **قيمة تقريبية** لأي نقطة قريبة منها على رسم الدالة

Linear approximation or **tangent line approximation**

$$f(x) \approx f(a) + f'(a)(x - a)$$

Example 1

Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

Solution

$$f'(x) = \tfrac{1}{2}(x+3)^{-1/2}$$

$$f'(1) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$f(1) = \sqrt{4} = 2$$

$$\therefore L(x) = 2 + \frac{1}{4}(x-1)$$

$$= 2 + \frac{1}{4}x - \frac{1}{4} = \frac{x}{4} + \frac{7}{4}$$

$$\therefore f(x) \approx \frac{x}{4} + \frac{7}{4} \quad \text{for } x \text{ near } 1$$

$$\sqrt{3.98} \approx \frac{0.98}{4} + \frac{7}{4} = \frac{7.98}{4} = 1.995$$

$$\sqrt{4.05} \approx \frac{1.05}{4} + \frac{7}{4} = \frac{8.05}{4} = 2.0125$$

Example 2

Estimate $\sqrt{99.8}$.

Solution

$f(x) = \sqrt{x}$, $a = 100$, $f(a) = \sqrt{100} = 10$

$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$

$f(x) \approx f(a) + f'(a)(x-a)$

$\sqrt{99.8} \approx 10 + \frac{1}{20}(99.8 - 100)$

$\approx 10 - \frac{0.2}{20}$

$\approx 10 - 0.01 \approx 9.99$

Example 3

Approximate $(2.001)^5$.

Solution

$f(x) = x^5$, $a = 2$, $f(a) = 2^5 = 32$

$f'(x) = 5x^4$, $f'(2) = 5 \cdot 2^4 = 80$

$f(x) \approx f(a) + f'(a)(x-a)$

$(2.001)^5 \approx 32 + 80(2.001 - 2)$

$\approx 32 + 0.08$

≈ 32.08

Example 4

Use linear approximation to show that if $x \approx 1$, then $\sqrt{x} \approx \frac{1}{2}(x+1)$

Solution

$$f(x) = \sqrt{x} \Rightarrow f(1) = 1$$

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(1) = \frac{1}{2}$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$\sqrt{x} \approx 1 + \frac{1}{2}(x-1)$$

$$\approx 1 + \frac{1}{2}x - \frac{1}{2}$$

$$\approx \frac{1}{2} + \frac{1}{2}x$$

$$\approx \frac{1}{2}(x+1)$$

- Differentials

مشتقة x

$$dy = f'(x)\, dx$$

y مشتقة

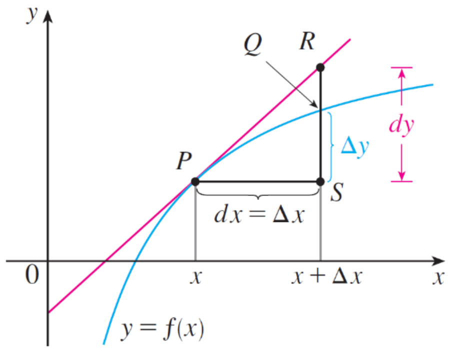
dy : قيمة تغير y التقريبية

نتيجة تغير x

$$\Delta y = f(x + \Delta x) - f(x)$$

Δy : قيمة تغير y الدقيقة

نتيجة تغير x



Example 5

Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes from 2 to 2.05.

Solution

$$f(2) = 2^3 + 2^2 - 2 \cdot 2 + 1 = 9$$

$$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1$$
$$= 9.7176$$

$$\Delta y = f(2.05) - f(2) = 0.7176$$

$$f'(x) = 3x^2 + 2x - 2$$

$x = 2$

$dx = \Delta x = 0.05$

$$dy = f'(x)\, dx$$
$$= (3 \cdot 2^2 + 2 \cdot 2 - 2)\, 0.05$$
$$= 14 \times 0.05 = 0.7$$

Problems

- Find the linearization $L(x)$ of $f(x) = \sin x$ at $a = \pi/6$.

- Find the differential of each function.

(a) $y = xe^{-4x}$

(b) $y = \frac{1+2u}{1+3u}$

- Use a linear approximation to estimate the given number.

(a) $1/4.002$

(b) $\cos 29°$