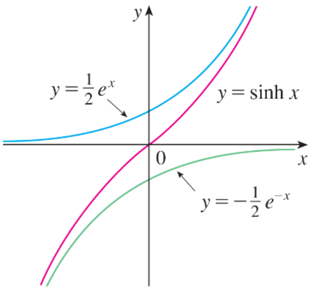
Section 3.11 – Hyperbolic Functions

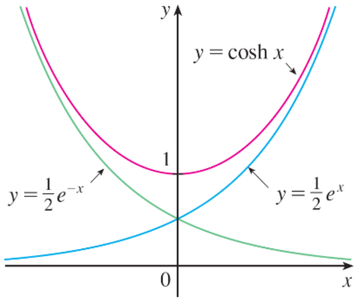
- Definition of the Hyperbolic Functions

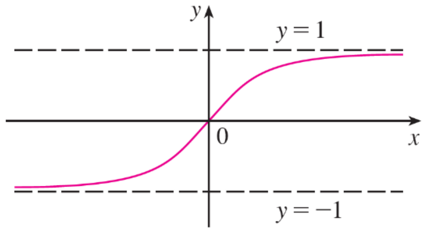
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$







Example 1

Use the definitions of the hyperbolic functions to find each of the following limits.

(a) $\lim\limits_{x\to\infty} \tanh x$ (b) $\lim\limits_{x\to-\infty} \tanh x$

Solution

(a) $\lim\limits_{x\to\infty} \tanh x = \lim\limits_{x\to\infty} \dfrac{\sinh x}{\cosh x}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})}$

$= \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \dfrac{e^{-x}}{e^{-x}}$

$= \lim\limits_{x\to\infty} \dfrac{1 - \cancel{e^{-2x}}^{\,0}}{1 + \cancel{e^{-2x}}^{\,0}} = 1$

- Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Example 2

Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and (b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

Solution

(a) $$\cosh^2 x - \sinh^2 x = \left(\frac{e^{x}+e^{-x}}{2}\right)^2 - \left(\frac{e^{x}-e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x}+2+e^{-2x}}{4} - \frac{e^{2x}-2+e^{-2x}}{4}$$

$$= \frac{4}{4} = 1$$

(b) From the identity $\cosh^2 x - \sinh^2 x = 1$

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

- Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Example 3

Prove $\frac{d}{dx}(\sinh x) = \cosh x$

Solution

$\sinh x = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$

$\frac{d}{dx}(\sinh x) = \frac{1}{2}e^{x} + \frac{1}{2}e^{x} = \cosh x$

Example 4

Find the derivative of the following function

(a) $y = \cosh\sqrt{x}$

(b) $y = \csc\pi + \cosh x^5$

Solution

(a) $y' = \frac{1}{2\sqrt{x}} \sinh\sqrt{x}$

(b) $y' = 0 + 5x^4 \cdot \sinh x$

$= 5x^4 \sinh x$

Problems

- Find the numerical value of each expression.

(a) $\tanh 0$

$$= \frac{(e^{0} - e^{-0})/\cancel{2}}{(e^{0} + e^{-0})/\cancel{2}} = \frac{1-1}{1+1} = 0$$

(b) $\tanh 1$

$$= \frac{(e^{1} - e^{-1})/\cancel{2}}{(e^{1} + e^{-1})/\cancel{2}} = \frac{e - \frac{1}{e}}{e + \frac{1}{e}} \cdot \frac{e}{e}$$

$$= \frac{e^2 - 1}{e^2 + 1}$$

(c) $\sinh 4$

$$= \frac{e^4 - e^{-4}}{2}$$

(d) $\sinh(\ln 4)$

$$= \frac{e^{\ln 4} - e^{-\ln 4}}{2}$$

$$= \frac{4 - e^{\ln 4^{-1}}}{2} = \frac{4 - 4^{-1}}{2}$$

$$= \frac{4 - \frac{1}{4}}{2} \cdot \frac{4}{4} = \frac{16 - 1}{8} = \frac{15}{8}$$

- Prove the identity

(a) $\cosh(-x) = \cosh x$

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2}$$

$$= \frac{e^{-x} + e^{x}}{2} = \cosh x$$

(b) $\cosh x - \sinh x = e^{-x}$

$$\cosh x - \sinh x = \frac{e^{x} + e^{-x}}{2} - \frac{e^{x} - e^{-x}}{2}$$

$$= \frac{\cancel{e^{x}} + e^{-x} - \cancel{e^{x}} + e^{-x}}{2}$$

$$= \frac{\cancel{2}e^{-x}}{\cancel{2}} = e^{-x}$$

- Differentiate

(a) $y = \coth\left(\frac{1}{x}\right) + \sinh(x)\tanh(4x^2)$

$$y' = -\frac{1}{x^2} \cdot -\operatorname{csch}^2\left(\frac{1}{x}\right) + \sinh(x) \cdot 8x \operatorname{sech}^2(4x^2) + \cosh x \cdot \tanh(4x^2)$$

$$= \frac{1}{x^2}\operatorname{csch}^2\left(\frac{1}{x}\right) + 8x \sinh x \cdot \operatorname{sech}^2(4x^2) + \cosh x \cdot \tanh(4x^2)$$

(b) $y = \sinh(x^2+1)^x$

$$y' = \frac{d}{dx}(x^2+1)^x \cdot \cosh(x^2+1)^x$$

$$r = (x^2+1)^x$$

$$\ln r = x \ln(x^2+1)$$

$$\frac{r'}{r} = x \cdot \frac{2x}{x^2+1} + \ln(x^2+1)$$

$$r' = \left[\frac{2x^2}{x^2+1} + \ln(x^2+1)\right] \cdot (x^2+1)^x$$

$$\therefore y' = \left[\frac{2x^2}{x^2+1} + \ln(x^2+1)\right](x^2+1)^x \cdot \cosh(x^2+1)^x$$