Section 3.4 – The Chain Rule

- Composite functions الدوال المركبة

Given $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$

$$F(x) = f \circ g = f(g(x)) = \sqrt{x^2 + 1}$$

$$f(u) = \sqrt{u} \qquad u = g(x) = x^2 + 1$$

- The Chain Rule: قاعدة اشتقاق الدوال المركبة

In prime notation

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$= f'(u) \cdot u'$$

In Leibniz notation

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 1

Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$

Solution

$$g(x) = x^2 + 1, \quad F(x) = f(g(x)) = \sqrt{g(x)}$$

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$= \frac{1}{2}[g(x)]^{-1/2} \cdot (2x)$$

$$= \frac{1}{2}[x^2 + 1]^{-1/2} (2x) = \frac{x}{\sqrt{x^2 + 1}}$$

Example 2

Differentiate:

(a) $y = \sin(x^2)$ (b) $y = \sin^2 x$

Solution

(a) $y = \sin u$, $u(x) = x^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \cos u \cdot 2x$$

$$= 2x \cos(x^2)$$

(b) $y = u^2$, $u(x) = \sin x$

$$\frac{dy}{du} = 2u \cdot \cos x$$

$$= 2 \cdot \sin x \cdot \cos x$$

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- Trigonometric Functions

$$\frac{d}{dx}\sin(f(x)) = f'(x) \cdot \cos(f(x))$$

$$\frac{d}{dx}\csc f(x) = f'(x) \cdot -\cot(f(x))\csc(f(x))$$

$$\frac{d}{dx}\cos(f(x)) = f'(x) \cdot -\sin(f(x))$$

$$\frac{d}{dx}\sec(f(x)) = f'(x) \cdot \tan(f(x))\sec(f(x))$$

$$\frac{d}{dx}\tan(f(x)) = f'(x) \cdot \sec^2(f(x))$$

$$\frac{d}{dx}\cot(f(x)) = f'(x) \cdot -\csc^2(f(x))$$

- The Power Rule with the Chain Rule

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

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Example 3

Differentiate $y = (x^3 - 1)^{100}$

Solution

$$y' = 100(x^3 - 1)^{99}(3x^2)$$

$$= 300x^2(x^3 - 1)^{99}$$

Example 4

Differentiate $y = (2x + 1)^5(x^3 - x + 1)^4$

Solution

$$u = (2x+1)^5, \quad v = (x^3 - x + 1)^4$$

$$y' = uv' + u'v \qquad \text{product rule}$$

$$y' = (2x+1)^5 \cdot 4(x^3 - x + 1)^3(3x^2 - 1)$$

$$+ 5(2x+1)^4 \cdot 2(x^3 - x + 1)^4$$

Example 5

Find $f'(x)$ if $f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$

Solution

$$f(x) = (x^2+x+1)^{-1/3}$$

$$f'(x) = -\tfrac{1}{3}(x^2+x+1)^{-4/3} \cdot (2x+1)$$

$$= -\frac{2x+1}{3\sqrt[3]{(x^2+x+1)^4}}$$

Example 6

Find the derivative of the function

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9$$

Solution

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \left(\frac{(2t+1)\cdot 1 - (t-2)\cdot 2}{(2t+1)^2}\right)$$

$$= 9\,(t-2)^8\,[\,2t+1 - 2t$$

$$= 45\,\frac{(t-2)^8}{(2t+1)^{10}}$$

- Exponential Function

$$\frac{d}{dx}\left(b^{f(x)}\right) = f'(x) \cdot b^{f(x)} \ln b$$

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$$\frac{d}{dx} e^{f(x)} = f'(x)\, e^{f(x)}$$

Example 7

Find the derivative of the function $y = 2^{4x}$

Solution

$$y' = 4 \cdot 2^{4x} \cdot \ln 2$$

$$= 4 \cdot 16^{x} \ln 2$$

Example 8

Differentiate $y = e^{\cos 3\theta}$.

Solution

$$y' = \underbrace{-3\sin 3\theta}_{f'(x)} \cdot e^{\overbrace{\cos 3\theta}^{f(x)}}$$

Problems

- Write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$]. Then find the derivative dy/dx.

(a) $\sqrt{4+3x}$

$u = 4 + 3x \,, \quad y = \sqrt{u}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \tfrac{1}{2} u^{-1/2} (3)$$

$$= \frac{3}{2\sqrt{4+3x}}$$

(b) $y = \tan(\sin x)$

$u = \sin x \,, \quad y = \tan u$

$$\frac{dy}{dx} = \sec^2 u \cdot \cos x$$

$$= \sec^2(\sin x) \cdot \cos x$$

- **Find the derivative of the function**

(a) $f(x) = \frac{1}{\sqrt[3]{x^2-1}}$

$f(x) = (x^2-1)^{-1/3}$

$f'(x) = -\frac{1}{3}(x^2-1)^{-4/3} \cdot (2x)$

$= -\frac{2x}{3\sqrt[3]{(x^2-1)^4}}$

(b) $g(\theta) = \cos^2\theta$

$g'(\theta) = 2\cos\theta \cdot -\sin\theta$

(c) $f(t) = e^{at}\sin bt$

$f' = e^{at} \cdot b\cos bt + ae^{at}\sin bt$

(d) $y = \left(x + \frac{1}{x}\right)^5$

$$y' = 5\left(x + \frac{1}{x}\right)^4 \cdot \left(1 - \frac{1}{x^2}\right)$$

(e) $f(z) = e^{z/(z-1)}$

$$f'(z) = \frac{(z-1)\cdot 1 - z \cdot 1}{(z-1)^2} e^{z/(z-1)}$$

$$= \frac{z - 1 - z}{(z-1)^2} e^{z/(z-1)}$$

$$= \frac{-e^{z/(z-1)}}{(z-1)^2}$$

(f) $H(r) = \frac{(r^2-1)^3}{(2r+1)^5}$

$$H'(r) = \frac{(2r+1)^5 \cdot 3(r^2-1)^2 \cdot 2r - (r^2-1)^3 \cdot 5(2r+1)^4 \cdot 2}{\left((2r+1)^5\right)^2}$$

$$= \frac{6r(2r+1)^5(r^2-1)^2 - 10(r^2-1)^3(2r+1)^4}{(2r+1)^{10}}$$

(g) $y = \cos\sqrt{\sin(\tan \pi x)}$

$y = \cos u \quad , \quad u = \sqrt{v} \quad , \quad v = \sin w \quad , \quad w = \tan \pi x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$$

$$= -\sin u \cdot \tfrac{1}{2} v^{-1/2} \cdot \cos w \cdot \pi \sec^2 \pi x$$

$$= -\sin\sqrt{\sin(\tan \pi x)} \cdot \tfrac{1}{2}[\sin(\tan \pi x)]^{-1/2}$$

$$\cos(\tan \pi x) \cdot \pi \sec^2 \pi x$$

(h) $y = [x + (x + \sin^2 x)^3]^4$

$$y' = 4[x + (x + \sin^2 x)^3]^3$$

$$\cdot [1 + 3(x + \sin^2 x)^2 \cdot (1 + 2\sin x \cos x)]$$

- At what point on the curve $y = \sqrt{1+2x}$ is the tangent line perpendicular to the line $6x + 2y = 1$?

$$2y = -6x + 1$$

$$y = -3x + \frac{1}{2}$$

$$m_1 = -3$$

$$m_2 = -\frac{1}{m_1} = \frac{1}{3}$$ perpendicular

$$y' = \frac{1}{2}(1+2x)^{-1/2} \cdot 2$$

$$= (1+2x)^{-1/2} = \frac{1}{\sqrt{1+2x}}$$

$$\therefore \frac{1}{\sqrt{1+2x}} = \frac{1}{3}$$

$$\sqrt{1+2x} = 3$$

$$1 + 2x = 9$$

$$2x = 8$$

$$x = 4$$

$$y(4) = \sqrt{1+2(4)} = \sqrt{9} = 3$$

the point is $(4, 3)$

- Let $r(x) = f\left(g\left(h(x)\right)\right)$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

$r'(x) = f'(g) \cdot g'(h) \cdot h'(x)$

$r'(1) = h'(1) \cdot g'(h(1)) \cdot f'(g(2))$

$= 4 \cdot g'(2) \cdot f'(3)$

$= 4 \cdot 5 \cdot 6 = 120$