Section 3.4 – The Chain Rule

- Composite functions الدوال المركبة

Given $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$

$$F(x) = f \circ g = f\big(g(x)\big) = \sqrt{x^2 + 1}$$

$$f(u) = \sqrt{u} \qquad u = g(x) = x^2 + 1$$

- The Chain Rule: قاعدة اشتقاق الدوال المركبة

In prime notation

$$F'(x) = f'\big(g(x)\big) \cdot g'(x)$$

$$= f'(u) \cdot u'$$

In Leibniz notation

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 1

Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$

Solution

$$g(x) = x^2 + 1, \quad F(x) = f(g(x)) = \sqrt{g(x)}$$

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$= \frac{1}{2}[g(x)]^{-1/2} \cdot (2x)$$

$$= \frac{1}{2}[x^2 + 1]^{-1/2} (2x) = \frac{x}{\sqrt{x^2 + 1}}$$

Example 2

Differentiate:

(a) $y = \sin(x^2)$ (b) $y = \sin^2 x$

Solution

(a) $y = \sin u$, $u(x) = x^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \cos u \cdot 2x$$

$$= 2x \cos(x^2)$$

(b) $y = u^2$, $u(x) = \sin x$

$$\frac{dy}{du} = 2u \cdot \cos x$$

$$= 2 \cdot \sin x \cdot \cos x$$

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- Trigonometric Functions

$$\frac{d}{dx}\sin(f(x)) = f'(x) \cdot \cos(f(x))$$

$$\frac{d}{dx}\csc f(x) = f'(x) \cdot -\cot(f(x))\csc(f(x))$$

$$\frac{d}{dx}\cos(f(x)) = f'(x) \cdot -\sin(f(x))$$

$$\frac{d}{dx}\sec(f(x)) = f'(x) \cdot \tan(f(x))\sec(f(x))$$

$$\frac{d}{dx}\tan(f(x)) = f'(x) \cdot \sec^2(f(x))$$

$$\frac{d}{dx}\cot(f(x)) = f'(x) \cdot -\csc^2(f(x))$$

- The Power Rule with the Chain Rule

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

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Example 3

Differentiate $y = (x^3 - 1)^{100}$

Solution

$$y' = 100(x^3-1)^{99}(3x^2)$$

$$= 300x^2(x^3-1)^{99}$$

Example 4

Differentiate $y = (2x+1)^5(x^3 - x + 1)^4$

Solution

$u = (2x+1)^5 \quad , \quad v = (x^3 - x + 1)^4$

$y' = uv' + u'v$ product rule

$$y' = (2x+1)^5 \cdot 4(x^3-x+1)^3(3x^2-1)$$

$$+ 5(2x+1)^4 \cdot 2(x^3-x+1)^4$$

Example 5

Find $f'(x)$ if $f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$

Solution

$$f(x) = (x^2+x+1)^{-1/3}$$

$$f'(x) = -\tfrac{1}{3}(x^2+x+1)^{-4/3} \cdot (2x+1)$$

$$= -\frac{2x+1}{3\sqrt[3]{(x^2+x+1)^4}}$$

Example 6

Find the derivative of the function

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9$$

Solution

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \left(\frac{(2t+1)\cdot 1 - (t-2)\cdot 2}{(2t+1)^2}\right)$$

$$= 9\,(t-2)^8\,[\,\cancel{2t}+1 - \cancel{2t}$$

$$= 45\,\frac{(t-2)^8}{(2t+1)^{10}}$$

- Exponential Function

$$\frac{d}{dx}\left(b^{f(x)}\right) = f'(x) \cdot b^{f(x)} \ln b$$

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$$\frac{d}{dx} e^{f(x)} = f'(x)\, e^{f(x)}$$

Example 7

Find the derivative of the function $y = 2^{4x}$

Solution

$$y' = 4 \cdot 2^{4x} \cdot \ln 2$$

$$= 4 \cdot 16^{x} \ln 2$$

Example 8

Differentiate $y = e^{\cos 3\theta}$.

Solution

$$y' = \underbrace{-3\sin 3\theta}_{f'(x)} \cdot e^{\overbrace{\cos 3\theta}^{f(x)}}$$

Problems

- Write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$]. Then find the derivative dy/dx.

(a) $\sqrt{4 + 3x}$

(b) $y = \tan(\sin x)$

- Find the derivative of the function

(a) $f(x) = \frac{1}{\sqrt[3]{x^2-1}}$

(b) $g(\theta) = \cos^2\theta$

(c) $f(t) = e^{at}\sin bt$

(d) $y = \left(x + \frac{1}{x}\right)^5$

(e) $f(z) = e^{z/(z-1)}$

(f) $H(r) = \frac{\left(r^2-1\right)^3}{(2r+1)^5}$

(g) $y = \cos\sqrt{\sin(\tan \pi x)}$

(h) $y = [x + (x + \sin^2 x)^3]^4$

- At what point on the curve $y = \sqrt{1 + 2x}$ is the tangent line perpendicular to the line $6x + 2y = 1$?

- Let $r(x) = f\left(g(h(x))\right)$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.