Section 3.5 – Implicit Differentiation

- نستخدم implicit differentiation إذا كانت العلاقة بين المتغيرين غير مباشرة

Explicit علاقة مباشرة

$$y = \sqrt{x^3 + 1}$$

نقدر نعزل y في طرف

Implicit علاقة غير مباشرة

$$x^3 + y^3 = 6xy$$

لا يمكن عزل y في طرف

Example 1

(a) If $x^2 + y^2 = 25$, find y'.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3,4)$.

Solution

(a) $2x + 2y \cdot y' = 0$

$2yy' = -2x$

$y' = \frac{-2x}{2y} = -\frac{x}{y}$

(b) $y - y_0 = m(x - x_0)$

$m = y'_{(3,4)} = -\frac{3}{4}$

$y - 4 = -\frac{3}{4}(x - 3)$

الخلاصة

عند اشتقاق y نطبق قواعد الاشتقاق العادية، ثم نضيف y'

مثلا:

$$2y^3 \xrightarrow{\frac{d}{dx}} 6y^2 \cdot y'$$

$$y \xrightarrow{\frac{d}{dx}} 1 \cdot y' = y'$$

Example 2

(a) Find y' if $x^3 + y^3 = 6xy$.
(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3,3)$.

Solution

(a) $3x^2 + 3y^2 \cdot y' = 6x \cdot y' + 6y$

$$3y^2 y' - 6x y' = 6y - 3x^2$$

$$y^2 y' - 2x y' = 2y - x^2$$

$$y'(y^2 - 2x) = 2y - x^2$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

(b) $y - y_0 = m(x - x_0)$

$$m = y'_{(3,3)} = \frac{2(3) - 3^2}{3^2 - 2(3)} = -1$$

$$y - 3 = -(x - 3)$$

Example 3

Find y' if $\sin(x + y) = y^2 \cos x$.

Solution

$$\cos(x+y) \cdot (1 + y') = y^2 \cdot -\sin x + 2y \cdot y' \cos x$$

$$y' \cos(x+y) - 2y y' \cos x = -y^2 \sin x - \cos(x+y)$$

$$y'(\cos(x+y) - 2y\cos x) = -y^2 \sin x - \cos(x+y)$$

$$y' = \frac{-y^2 \sin x - \cos(x+y)}{\cos(x+y) - 2y\cos x}$$

Example 4

Find y'' if $x^4 + y^4 = 16$.

Solution

$$4x^3 + 4y^3 \cdot y' = 0$$

$$4y^3 y' = -4x^3$$

$$y' = -\frac{4x^3}{4y^3} = -\frac{x^3}{y^3}$$

$$y'' = -\frac{y^3 \cdot 3x^2 - x^3 \cdot 3y^2 y'}{(y^3)^2}$$

$$y'' = -\frac{3x^2 y^3 - 3x^3 y^2 \left(-\frac{x^3}{y^3}\right)}{y^6}$$

$$y'' = -\frac{3x^2 y^3 + \frac{3x^6}{y}}{y^6} \cdot \frac{y}{y}$$

$$y'' = -\frac{3x^2 y^4 + 3x^6}{y^7}$$

- Derivatives of Inverse Trigonometric Functions:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

Example 5

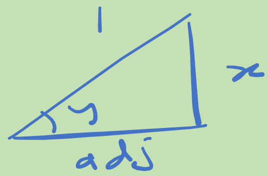
Show that $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$

Solution

Let $\sin^{-1}(x) = y$

$\sin y = x$

where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



$\sin y = \frac{opp}{hyp} = \frac{x}{1}$

$adj = \sqrt{1-x^2}$

by differentiation

$y' \cos y = 1$

$y' = \frac{1}{\cos y}$

$\cos y = \frac{adj}{hyp} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

$\therefore \; y' = \frac{1}{\sqrt{1-x^2}}$

Example 6

Differentiate:

(a) $y = \frac{1}{\sin^{-1} x}$ (b) $f(x) = x \arctan \sqrt{x}$

Solution

(a)
$$y' = \frac{\cancel{\sin^{-1} x \cdot (0)} - 1 \cdot \frac{1}{\sqrt{1-x^2}}}{(\sin^{-1} x)^2}$$

$$= -\frac{1}{(\sin^{-1} x)^2 \sqrt{1-x^2}}$$

حل آخر

$$y = (\sin^{-1} x)^{-1}$$

$$y' = -(\sin^{-1} x)^{-2} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= -\frac{1}{(\sin^{-1} x)^2 \sqrt{1-x^2}}$$

(b)
$$f'(x) = x \cdot \tfrac{1}{2} x^{-1/2} \frac{1}{1+(\sqrt{x})^2} + \tan^{-1} \sqrt{x}$$

$$= \frac{x}{2\sqrt{x}\,(1+x)} + \tan^{-1} \sqrt{x}$$

Problems

For the equation $\frac{2}{x} - \frac{1}{y} = 4$

(a) Find y' by implicit differentiation.

$$2x^{-1} - y^{-1} = 4$$

$$-2x^{-2} + y^{-2} \cdot y' = 0$$

$$\frac{y'}{y^2} = \frac{2}{x^2}$$

$$y' = \frac{2y^2}{x^2}$$

(b) Solve the equation explicitly for y and differentiate to get y' in terms of x.

$$-\frac{1}{y} = 4 - \frac{2}{x}$$

$$-\frac{1}{y} = \frac{4x-2}{x}$$

$$y = \frac{-x}{4x-2}$$

$$y' = \frac{(4x-2)\cdot -1 - (-x)(4)}{(4x-2)^2}$$

$$= \frac{\cancel{-4x} + 2 + \cancel{4x}}{(4x-2)^2} = \frac{2}{(4x-2)^2}$$

(c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

$$y' = \frac{2y^2}{x^2} = \frac{2\left(-x/4x-2\right)^2}{x^2} = \frac{2\cancel{x^2}}{\cancel{x^2}(4x-2)^2}$$

- Find dy/dx by implicit differentiation.

(a) $2x^2 + xy - y^2 = 2$

$$4x + xy' + y - 2yy' = 0$$

$$xy' - 2yy' = -4x - y$$

$$y' = \frac{-4x - y}{x - 2y}$$

(b) $x^3 - xy^2 + y^3 = 1$

$$3x^2 - x \cdot 2y \cdot y' + 3y^2 y' = 0$$

$$-2xyy' + 3y^2 y' = -3x^2$$

$$y' = \frac{-3x^2}{-2xy + 3y^2}$$

(c) $x^4(x + y) = y^2(3x - y)$

$$x^5 + x^4 y = 3xy^2 - y^3$$

$$5x^4 + x^4 \cdot y' + 4x^3 y = 3x \cdot 2yy' + 3y^2 - 3y^2 y'$$

$$x^4 y' - 6xyy' + 3y^2 y' = 3y^2 - 5x^4 - 4x^3 y$$

$$y' = \frac{3y^2 - 5x^4 - 4x^3 y}{x^4 - 6xy + 3y^2}$$

(d) $e^{x/y} = x - y$

$$e^{x/y} \cdot \frac{xy' - y}{y^2} = 1 - y'$$

$$\frac{xy' e^{x/y}}{y^2} - \frac{\cancel{y} e^{x/y}}{y^{\cancel{2}}} = 1 - y'$$

$$\frac{x y' e^{x/y}}{y} + y' = 1 + \frac{e^{x/y}}{y}$$

$$y' = \frac{1 + e^{x/y}/y}{x e^{x/y}/y + 1}$$

(e) $\tan^{-1}(x^2y) = x + xy^2$

$$(x^2y' + 2xy)\cdot\frac{1}{1+(x^2y)^2} = 1 + x\cdot 2yy' + y^2$$

$$\frac{x^2y'}{1+x^4y^2} + \frac{2xy}{1+x^4y^2} = 1 + 2xyy' + y^2$$

$$\frac{x^2}{1+x^4y^2}y' - 2xyy' = 1 + y^2 - \frac{2xy}{1+x^4y^2}$$

$$y' = \frac{1+y^2 - 2xy/(1+x^4y^2)}{x^2/(1+x^4y^2) - 2xy}$$

(f) $\sin(xy) = \cos(x+y)$

$$(xy' + y)\cos(xy) = -\sin(x+y)\cdot(1+y')$$

$$xy'\cos(xy) + y\cos(xy) = -\sin(x+y) - y'\sin(x+y)$$

$$xy'\cos(xy) + y'\sin(x+y) = -\sin(x+y) - y\cos(xy)$$

$$y' = \frac{-\sin(x+y) - y\cos(xy)}{x\cos(xy) + \sin(x+y)}$$

- **If $f(x) + x^2[f(x)]^3 = 10$ and $f(1) = 2$, find $f'(1)$.**

$$f'(x) + x^2 \cdot 3\,[f(x)]^2 \cdot f'(x) + 2x\,[f(x)]^3 = 0$$

$$f'(x)\left[1 + 3x^2 [f(x)]^2\right] = -2x\,[f(x)]^3$$

$$f'(x) = \frac{-2x\,[f(x)]^3}{1 + 3x^2 [f(x)]^2}$$

$$f'(1) = \frac{-2 \cdot 1 \cdot 2^3}{1 + 3 \cdot 1^2 \cdot 2^2} = \frac{-16}{13}$$

- **If $g(x) + x \sin g(x) = x^2$, find $g'(0)$.**

$$g'(x) + x \cdot \cos g(x) \cdot g'(x) + \sin g(x) = 2x$$

$$g'(x)\left[1 + x \cos g(x)\right] = 2x - \sin g(x)$$

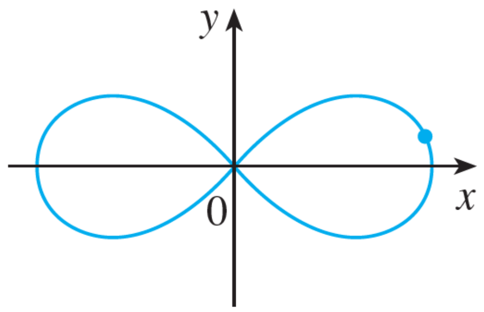
$$g'(x) = \frac{2x - \sin g(x)}{1 + x \cos g(x)}$$

$$g'(0) = \frac{\cancel{2 \cdot 0} - \sin g(0)}{1 + \cancel{0 \cos g(\ )}}$$

$$g'(0) = -\sin g(0)$$

- Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$2(x^2 + y^2)^2 = 25(x^2 - y^2), \quad (3, 1)$$



$$4(x^2+y^2)\cdot(2x+2y\cdot y') = 25(2x - 2yy')$$

$$8x^3 + 8x^2yy' + 8xy^2 + 8y^3y' = 50x - 50yy'$$

$$8x^2yy' + 8y^3y' + 50yy' = 50x - 8x^3 - 8xy^2$$

$$y'(8x^2y + 8y^3 + 50y) = 50x - 8x^3 - 8xy^2$$

$$y' = \frac{50x - 8x^3 - 8xy^2}{8x^2y + 8y^3 + 50y} = \frac{25x - 4x(x^2+y^2)}{25y + 4y(x^2+y^2)}$$

$$y - y_0 = m(x - x_0)$$

$$m = y'_{(3,1)} = \frac{25(3) - 4\cdot 3(3^2+1^2)}{25(1) - 4\cdot 1(3^2+1^2)}$$

$$= \frac{75-120}{25-40} = 3$$

$$\therefore \quad y - 1 = 3(x - 3)$$

- Find the derivative of the function. Simplify where possible.

(a) $y = \sqrt{\tan^{-1} x}$

$$y' = \frac{1}{2}(\tan^{-1} x)^{-1/2} \cdot \frac{1}{1+x^2}$$

$$= \frac{1}{2\sqrt{\tan^{-1} x}\,(1+x^2)}$$

(b) $y = \sin^{-1}(2x+1)$

$$y' = 2 \cdot \frac{1}{\sqrt{1-(2x+1)^2}}$$

$$= \frac{2}{\sqrt{\cancel{1} - 4x^2 - 4x - \cancel{1}}}$$

$$= \frac{2}{\sqrt{-4x(x+1)}}$$

$$= \frac{\cancel{2}}{\cancel{2}\sqrt{-x(x+1)}} = \frac{1}{\sqrt{-x(x+1)}}$$

(c) $g(x) = \arccos\sqrt{x}$

$$g'(x) = \tfrac{1}{2} x^{-1/2} \cdot - \frac{1}{\sqrt{1-(\sqrt{x})^2}}$$

$$= - \frac{1}{2\sqrt{x}\,\sqrt{1-x}}$$

(d) $y = \tan^{-1}\left(x - \sqrt{1+x^2}\right)$

$$y' = \left(1 - \tfrac{1}{\cancel{2}}(1+x^2)^{-1/2} \cdot \cancel{2}x\right) \cdot \frac{1}{1+\left(x-\sqrt{1+x^2}\right)^2}$$

$$= \left(1 - \frac{x}{\sqrt{1+x^2}}\right) \cdot \frac{1}{1+x^2 - 2x\sqrt{1+x^2} + 1+x^2}$$

$$= \frac{\sqrt{1+x^2} - x}{\sqrt{1+x^2}} \cdot \frac{1}{2\left(x^2+1-x\sqrt{1+x^2}\right)}$$

$$= \frac{\sqrt{1+x^2} - x}{2\left[(x^2+1)\sqrt{1+x^2} - x(1+x^2)\right]}$$

$$= \frac{\cancel{\sqrt{1+x^2} - x}}{2(x^2+1)\cancel{\left[\sqrt{1+x^2} - x\right]}} = \frac{1}{2(x^2+1)}$$