Section 3.5 – Implicit Differentiation

- نستخدم implicit differentiation إذا كانت العلاقة بين المتغيرين غير مباشرة

Explicit علاقة مباشرة

$y = \sqrt{x^3 + 1}$

نقدر نعزل y في طرف

Implicit علاقة غير مباشرة

$x^3 + y^3 = 6xy$

لا يمكن عزل y في طرف

Example 1

(a) If $x^2 + y^2 = 25$, find y'.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3,4)$.

Solution

(a) $2x + 2y \cdot y' = 0$

$2yy' = -2x$

$y' = \frac{-2x}{2y} = -\frac{x}{y}$

(b) $y - y_0 = m(x - x_0)$

$m = y'_{(3,4)} = -\frac{3}{4}$

$y - 4 = -\frac{3}{4}(x - 3)$

الخلاصة

عند اشتقاق y نطبق قواعد الاشتقاق العادية، ثم نضيف y'

مثلا:

$$2y^3 \xrightarrow{\frac{d}{dx}} 6y^2 \cdot y'$$

$$y \xrightarrow{\frac{d}{dx}} 1 \cdot y' = y'$$

Example 2

(a) Find y' if $x^3 + y^3 = 6xy$.
(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3,3)$.

Solution

(a) $3x^2 + 3y^2 \cdot y' = 6x \cdot y' + 6y$

$$3y^2 y' - 6x y' = 6y - 3x^2$$

$$y^2 y' - 2x y' = 2y - x^2$$

$$y'(y^2 - 2x) = 2y - x^2$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

(b) $y - y_0 = m(x - x_0)$

$$m = y'_{(3,3)} = \frac{2(3) - 3^2}{3^2 - 2(3)} = -1$$

$$y - 3 = -(x - 3)$$

Example 3

Find y' if $\sin(x + y) = y^2 \cos x$.

Solution

$$\cos(x+y).(1+y') = y^2 . -\sin x + 2y \cdot y' \cos x$$

$$y' \cos(x+y) - 2y y' \cos x = -y^2 \sin x - \cos(x+y)$$

$$y'(\cos(x+y) - 2y\cos x) = -y^2 \sin x - \cos(x+y)$$

$$y' = \frac{-y^2 \sin x - \cos(x+y)}{\cos(x+y) - 2y\cos x}$$

Example 4

Find y'' if $x^4 + y^4 = 16$.

Solution

$$4x^3 + 4y^3 \cdot y' = 0$$

$$4y^3 y' = -4x^3$$

$$y' = -\frac{4x^3}{4y^3} = -\frac{x^3}{y^3}$$

$$y'' = -\frac{y^3 \cdot 3x^2 - x^3 \cdot 3y^2 y'}{(y^3)^2}$$

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6}$$

$$y'' = -\frac{3x^2y^3 + \frac{3x^6}{y}}{y^6} \cdot \frac{y}{y}$$

$$y'' = -\frac{3x^2y^4 + 3x^6}{y^7}$$

- Derivatives of Inverse Trigonometric Functions:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

Example 5

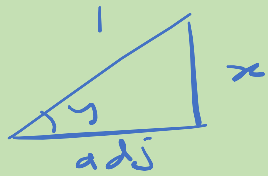
Show that $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$

Solution

Let $\sin^{-1}(x) = y$

$\sin y = x$

where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



$\sin y = \frac{opp}{hyp} = \frac{x}{1}$

$adj = \sqrt{1-x^2}$

by differentiation

$y' \cos y = 1$

$y' = \frac{1}{\cos y}$

$\cos y = \frac{adj}{hyp} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

$\therefore\ y' = \frac{1}{\sqrt{1-x^2}}$

Example 6

Differentiate:

(a) $y = \frac{1}{\sin^{-1} x}$ (b) $f(x) = x \arctan \sqrt{x}$

Solution

(a)

$$y' = \frac{\sin^{-1} x \cdot (0) - 1 \cdot \frac{1}{\sqrt{1-x^2}}}{(\sin^{-1} x)^2}$$

$$= -\frac{1}{(\sin^{-1} x)^2 \sqrt{1-x^2}}$$

حل آخر

$$y = (\sin^{-1} x)^{-1}$$

$$y' = -(\sin^{-1} x)^{-2} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= -\frac{1}{(\sin^{-1} x)^2 \sqrt{1-x^2}}$$

(b)

$$f'(x) = x \cdot \tfrac{1}{2} x^{-1/2} \frac{1}{1+(\sqrt{x})^2} + \tan^{-1}\sqrt{x}$$

$$= \frac{x}{2\sqrt{x}\,(1+x)} + \tan^{-1}\sqrt{x}$$

Problems

For the equation $\frac{2}{x} - \frac{1}{y} = 4$

(a) Find y' by implicit differentiation.

(b) Solve the equation explicitly for y and differentiate to get y' in terms of x.

(c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

- Find dy/dx by implicit differentiation.

(a) $2x^2 + xy - y^2 = 2$

(b) $x^3 - xy^2 + y^3 = 1$

(c) $x^4(x+y) = y^2(3x-y)$

(d) $e^{x/y} = x - y$

(e) $\tan^{-1}(x^2y) = x + xy^2$

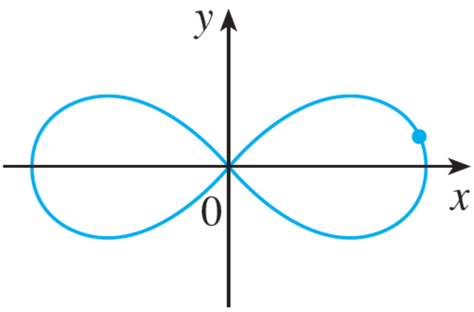
(f) $\sin(xy) = \cos(x + y)$

- If $f(x) + x^2[f(x)]^3 = 10$ and $f(1) = 2$, find $f'(1)$.

- If $g(x) + x \sin g(x) = x^2$, find $g'(0)$.

- Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$2(x^2 + y^2)^2 = 25(x^2 - y^2), \quad (3, 1)$$



- Find the derivative of the function. Simplify where possible.

(a) $y = \sqrt{\tan^{-1} x}$

(b) $y = \sin^{-1}(2x + 1)$

(c) $g(x) = \arccos\sqrt{x}$

(d) $y = \tan^{-1}\left(x - \sqrt{1 + x^2}\right)$