Section 3.6 – Derivatives of Logarithmic Functions

- Rules

$$\frac{d}{dx}(\log_b x) = \frac{1}{x\ln b} \quad , b \neq 1$$

$$\frac{d}{dx}(\log_b u(x)) = \frac{u'(x)}{u(x)\ln b} \quad , b \neq 1$$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

$$\frac{d}{dx}[\ln u(x)] = \frac{u'(x)}{u(x)}$$

Example 1

Show that $\frac{d}{dx}(\log_b x) = \frac{1}{x\ln b}$ for $b \neq 1$.

Solution

$y = \log_b x \Rightarrow b^y = x$

by differentiation

$b^y \cdot \ln b \cdot y' = 1$

$y' = \frac{1}{b^y \ln b}$

$= \frac{1}{x \ln b}$

تذكر

$\log_b x = y \Longleftrightarrow b^y = x$

Example 2

Find $\frac{d}{dx}\ln(x^3+1)$

Solution

$$\frac{d}{dx}\ln(x^3+1) = \frac{3x^2}{x^3+1}$$

Example 3

Differentiate

(a) $f(x) = \sqrt{\ln x}$

(b) $f(x) = \log(2+\sin x)$

(c) $g(x) = \ln|x|$

Solution

(a) $f'(x) = \frac{1}{2}(\ln x)^{-1/2} \cdot \frac{1}{x}$

$$= \frac{1}{2x\sqrt{\ln x}}$$

(b) $f'(x) = \dfrac{\cos x}{(2+\sin x)\ln 10}$

(c) $\ln|x| = \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases}$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}$$

$$g'(x) = \frac{1}{x} \qquad \forall x \neq 0$$

for all

- مشتقة دالة أس دالة Logarithmic Differentiation

$[f(x)]^{g(x)}$

$$y=[f(x)]^{g(x)}$$ ① نسميها بـ y

$$\ln y=\ln[f(x)]^{g(x)}$$ ② نأخذ ln للطرفين

$$\ln y=g(x)\ln[f(x)]$$ ③ نطبق قاعدة الأس

④ نشتق implicitly

$$\frac{y'}{y}=g(x)\cdot\frac{f'(x)}{f(x)}+g'(x)\ln[f(x)]$$

⑤ نضرب المعادلة في y

$$y'=y\left(g(x)\cdot\frac{f'(x)}{f(x)}+g'(x)\ln[f(x)]\right)$$

Example 4

Differentiate $y=x^{\sqrt{x}}$

Solution

$$\ln y=\sqrt{x}\ln x$$

$$\frac{y'}{y}=\sqrt{x}\cdot\frac{1}{x}+\frac{1}{2}x^{-1/2}\cdot\ln x$$

$$y'=\frac{y\sqrt{x}}{x}+\frac{y\ln x}{2\sqrt{x}}$$

لاحظ

يوجد 4 حالات لاشتقاق الأسس

1. الأساس والأس ثوابت constants

$$\frac{d}{dx}(e^3) = 0$$

2. الأساس متغير (معادلة) والأس ثابت

$$\frac{d}{dx}(x^2+1)^3 = 3(x^2+1)^2 \cdot 2x$$

3. الأساس ثابت والأس متغير

$$\frac{d}{dx}\left(5^{x^2+2x}\right) = 5^{x^2+2x}(\ln 5)\cdot(2x+2)$$

$$\frac{d}{dx}\left(e^{x^2+2x}\right) = e^{x^2+2x}(2x+2)$$

4. الأساس متغير والأس متغير (المثال السابق)

$$\frac{d}{dx}\left(3x^{x^2+2x}\right)$$

Logarithmic differentiation

Example 4

Differentiate $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$

Solution

$$y = \frac{x^{3/4}\,(x^2+1)^{1/2}}{(3x+2)^5}$$

كلها دوال وأسس

حلها بـ quotient rule مقرف

نستخدم logarithmic differentiation

$$\ln y = \ln x^{3/4} + \ln (x^2+1)^{1/2} - \ln (3x+2)^5$$

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln (x^2+1) - 5\ln (3x+2)$$

$$\frac{y'}{y} = \frac{3}{4}\,\frac{1}{x} + \frac{1}{\cancel{2}}\,\frac{\cancel{2}x}{x^2+1} - 5\,\frac{3}{3x+2}$$

$$y' = \frac{3y}{4x} + \frac{xy}{x^2+1} - \frac{15y}{3x+2}$$

تذكر

$$\ln(xy) = \ln x + \ln y$$

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

$$\ln(x^r) = r\ln x$$

- Special limits (The number *e* as a limit)

For $f(x) = \ln x \Rightarrow f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} \qquad \Rightarrow f'(1) = 1$

Using definition of differentiation

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - \cancel{f(1)}^{\,0}}{h}$$

$$1 = \lim_{h\to 0} \frac{\ln(1+h)}{h}$$

$$1 = \lim_{h\to 0} \frac{1}{h} \cdot \ln(1+h)$$

$$1 = \lim_{h\to 0} \ln(1+h)^{1/h}$$

$$e^1 = e^{\lim_{h\to 0} \ln(1+h)^{1/h}}$$

$$e = \lim_{h\to 0} e^{\ln(1+h)^{1/h}}$$

لأن e ثابت

$$\therefore \quad e = \lim_{h\to 0} (1+h)^{1/h}$$

$e^{\ln[f(x)]} = f(x)$ قاعدة

or $$e = \lim_{x\to \infty} \left(1 + \frac{1}{x}\right)^x$$

Problems

- Differentiate the function.

(a) $f(x) = \log(1 + \cos x)$

$\log x = \log_{10} x$

$$f'(x) = \frac{-\sin x}{(1+\cos x)\ln 10}$$

(b) $g(x) = \ln(xe^{-2x})$

$$g'(x) = \frac{x \cdot e^{-2x} \cdot -2 + e^{-2x}}{x e^{-2x}}$$

$$= \frac{-2x+1}{x}$$

(c) $F(t) = (\ln t)^2 \sin t$

$$F'(t) = (\ln t)^2 \cdot \cos t + 2(\ln t) \cdot \frac{1}{t} \cdot \sin t$$

$$= (\ln t)^2 \cos t + \frac{2\ln t}{t} \cdot \sin t$$

(d) $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

$$h'(x) = \frac{1 + \frac{1}{\cancel{2}}(x^2-1)^{-1/2} \cdot \cancel{2}x}{x + \sqrt{x^2-1}}$$

$$= \frac{1 + \frac{x}{\sqrt{x^2-1}}}{x + \sqrt{x^2-1}} \cdot \frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}$$

$$= \frac{\cancel{\sqrt{x^2-1} + x}}{\cancel{\left(x + \sqrt{x^2-1}\right)}\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$$

(e) $P(v) = \frac{\ln v}{1-v}$

$$P'(v) = \frac{(1-v)\frac{1}{v} - \ln v \cdot -1}{(1-v)^2}$$

$$= \frac{\frac{1-v}{v} + \ln v}{(1-v)^2} \cdot \frac{v}{v}$$

$$= \frac{1 - v + v\ln v}{v(1-v)^2}$$

(f) $y = \ln\left|1 + t - t^3\right|$

$$y' = \frac{1 - 3t^2}{1 + t - t^3}$$

(g) $y = \ln(e^{-x} + xe^{-x})$

$$y' = \frac{\cancel{-e^{-x}} + x \cdot -e^{-x} + \cancel{e^{-x}}}{e^{-x} + xe^{-x}}$$

$$= \frac{-xe^{-x}}{e^{-x} + xe^{-x}}$$

$$= \frac{-x\cancel{e^{-x}}}{\cancel{e^{-x}}(1+x)} = \frac{-x}{1+x}$$

(h) $g(x) = \ln \frac{x+1}{\sqrt{x-2}}$

$$g'(x) = \frac{\left[\sqrt{x-2} \cdot 1 - (x+1) \cdot \frac{1}{2}(x-2)^{-1/2} \cdot 1\right] / (\sqrt{x-2})^2}{(x+1)/\sqrt{x-2}}$$

$$= \frac{\left(\sqrt{x-2} - \frac{x+1}{2\sqrt{x-2}}\right)\sqrt{x-2}}{(x+1)(x-2)}$$

$$= \frac{x-2 - \frac{x+1}{2}}{(x+1)(x-2)} \cdot \frac{2}{2}$$

$$= \frac{2x-4-x-1}{2(x+1)(x-2)}$$

$$= \frac{x-5}{2(x+1)(x-2)}$$

(i) $H(z) = \ln\sqrt{\frac{a^2-z^2}{a^2+z^2}}$

$$H(z) = \ln \frac{(a^2-z^2)^{1/2}}{(a^2+z^2)^{1/2}}$$

$$= \ln(a^2-z^2)^{1/2} - \ln(a^2+z^2)^{1/2}$$

$$H'(z) = \frac{\frac{1}{\cancel{2}}(a^2-z^2)^{-1/2} \cdot (-\cancel{2}z)}{(a^2-z^2)^{1/2}} - \frac{\frac{1}{\cancel{2}}(a^2+z^2)^{-1/2}(\cancel{2}z)}{(a^2+z^2)^{1/2}}$$

$$= \frac{-z}{(a^2-z^2)^{1/2}(a^2-z^2)^{1/2}} - \frac{z}{(a^2+z^2)^{1/2}(a^2+z^2)^{1/2}}$$

$$= \frac{-z}{a^2-z^2} - \frac{z}{a^2+z^2}$$

(j) $y = \tan[\ln(ax+b)]$

$$y' = \frac{a}{ax+b} \cdot \sec^2[\ln(ax+b)]$$

- **Find y' and y''.**

$$y = \frac{\ln x}{1 + \ln x}$$

$$y' = \frac{(1+\ln x)\cdot\frac{1}{x} - \ln x\left(\frac{1}{x}\right)}{(1+\ln x)^2} \cdot \frac{x}{x}$$

$$= \frac{1 + \cancel{\ln x} - \cancel{\ln x}}{x(1+\ln x)^2} = \frac{1}{x(1+\ln x)^2}$$

- **Differentiate f and find the domain of f.**

$$f(x) = \frac{x}{1 - \ln(x-1)}$$

$$f'(x) = \frac{[1-\ln(x-1)]\cdot 1 - x\cdot -\frac{1}{x-1}}{[1-\ln(x-1)]^2} \cdot \frac{x-1}{x-1}$$

$$= \frac{(x-1)[1-\ln(x-1)] + x}{(x-1)[1-\ln(x-1)]^2}$$

$1-\ln(x-1) = 0$ and $x-1>0$

$\ln(x-1) = 1$ $\quad \ln \equiv \log_e$ $\quad x > 1$

$x - 1 = e$

$x = e+1$

$D = (1, 1+e) \cup (1+e, \infty)$

- Use logarithmic differentiation to find the derivative of the function.

(a) $y = \sqrt{x}e^{x^2}(x^2+1)^{10}$

$$\ln y = \ln\sqrt{x} + \ln e^{x^2} + \ln(x^2+1)^{10}$$

$$\ln y = \frac{1}{2}\ln x + x^2 \cancel{\ln e}^{1} + 10\ln(x^2+1)$$

$$\frac{y'}{y} = \frac{1}{2x} + 2x + 10\cdot\frac{2x}{x^2+1}$$

$$y' = \frac{y}{2x} + 2xy + \frac{20xy}{x^2+1}$$

(b) $y = x^x$

$$\ln y = \ln x^x$$

$$\ln y = x\ln x$$

$$\frac{y'}{y} = \cancel{x}\cdot\frac{1}{\cancel{x}} + \ln x$$

$$y' = y + y\ln x$$

(c) $y = x^{\sin x}$

$\ln y = \sin x \ln x$

$\frac{y'}{y} = \sin x \cdot \frac{1}{x} + \cos x \cdot \ln x$

$y' = \frac{y \sin x}{x} + y \cos x . \ln x$

(d) $y = \left(\sqrt{x}\right)^{x}$

$y = x^{\frac{1}{2}x}$

$\ln y = \frac{1}{2} x \ln x$

$\frac{y'}{y} = \frac{1}{2} \cancel{x} \cdot \frac{1}{\cancel{x}} + \frac{1}{2} \ln x$

$y' = \frac{1}{2} y + \frac{1}{2} y \ln x$