Section 3.6 – Derivatives of Logarithmic Functions

- Rules

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b} \quad , b \neq 1$$

$$\frac{d}{dx}(\log_b u(x)) = \frac{u'(x)}{u(x) \ln b} \quad , b \neq 1$$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

$$\frac{d}{dx}[\ln u(x)] = \frac{u'(x)}{u(x)}$$

Example 1

Show that $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$ for $b \neq 1$.

Solution

$y = \log_b x \Rightarrow b^y = x$

by differentiation

$b^y \cdot \ln b \cdot y' = 1$

$y' = \frac{1}{b^y \ln b}$

$= \frac{1}{x \ln b}$

تذكر

$\log_b x = y \iff b^y = x$

Example 2

Find $\frac{d}{dx}\ln(x^3+1)$

Solution

$$\frac{d}{dx}\ln(x^3+1) = \frac{3x^2}{x^3+1}$$

Example 3

Differentiate

(a) $f(x) = \sqrt{\ln x}$

(b) $f(x) = \log(2+\sin x)$

(c) $g(x) = \ln|x|$

Solution

(a) $f'(x) = \frac{1}{2}(\ln x)^{-1/2} \cdot \frac{1}{x}$

$$= \frac{1}{2x\sqrt{\ln x}}$$

(b) $f'(x) = \frac{\cos x}{(2+\sin x)\ln 10}$

(c) $\ln|x| = \begin{cases} \ln x & x \geq 0 \\ \ln(-x) & x < 0 \end{cases}$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}$$

$$f'(x) = \frac{1}{x} \quad \forall x \neq 0$$

$\forall$ → for all

- مشتقة دالة أس دالة Logarithmic Differentiation

$[f(x)]^{g(x)}$

① نسميها بـ y

$$y = [f(x)]^{g(x)}$$

② نأخذ ln للطرفين

$$\ln y = \ln [f(x)]^{g(x)}$$

③ نطبق قاعدة الأس

$$\ln y = g(x) \ln[f(x)]$$

④ نشتق implicitly

$$\frac{y'}{y} = g(x) \cdot \frac{f'(x)}{f(x)} + g'(x) \ln[f(x)]$$

⑤ نضرب المعادلة في y

$$y' = y \left(g(x) \cdot \frac{f'(x)}{f(x)} + g'(x) \ln[f(x)] \right)$$

Example 4

Differentiate $y = x^{\sqrt{x}}$

Solution

$$\ln y = \sqrt{x} \ln x$$

$$\frac{y'}{y} = \sqrt{x} \cdot \frac{1}{x} + \frac{1}{2} x^{-1/2} \cdot \ln x$$

$$y' = \frac{y\sqrt{x}}{x} + \frac{y \ln x}{2\sqrt{x}}$$

لاحظ

يوجد 4 حالات لاشتقاق الأسس

1. الأساس والأس ثوابت constants

$$\frac{d}{dx}(e^3) = 0$$

2. الأساس متغير (معادلة) والأس ثابت

$$\frac{d}{dx}(x^2+1)^3 = 3(x^2+1)^2 \cdot 2x$$

3. الأساس ثابت والأس متغير

$$\frac{d}{dx}\left(5^{x^2+2x}\right) = 5^{x^2+2x}(\ln 5) \cdot (2x+2)$$

$$\frac{d}{dx}\left(e^{x^2+2x}\right) = e^{x^2+2x}(2x+2)$$

4. الأساس متغير والأس متغير (المثال السابق)

$$\frac{d}{dx}\left(3x^{x^2+2x}\right)$$

Logarithmic differentiation

Example 4

Differentiate $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$

Solution

$$y = \frac{x^{3/4}(x^2+1)^{1/2}}{(3x+2)^5}$$

كلها دوال وأسس

حلها بـ quotient rule مقرف

نستخدم logarithmic differentiation

$$\ln y = \ln x^{3/4} + \ln(x^2+1)^{1/2} - \ln(3x+2)^5$$

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

$$\frac{y'}{y} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

$$y' = \frac{3y}{4x} + \frac{xy}{x^2+1} - \frac{15y}{3x+2}$$

تذكر

$$\ln(xy) = \ln x + \ln y$$

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

$$\ln(x^r) = r\ln x$$

- Special limits (The number *e* as a limit)

For $f(x) = \ln x \Rightarrow f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

Using definition of differentiation

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)^{\,0}}{h}$$

$$1 = \lim_{h \to 0} \frac{\ln(1+h)}{h}$$

$$1 = \lim_{h \to 0} \frac{1}{h} \cdot \ln(1+h)$$

$$1 = \lim_{h \to 0} \ln(1+h)^{1/h}$$

$$e^{1} = e^{\lim_{h \to 0} \ln(1+h)^{1/h}}$$

$$e = \lim_{h \to 0} e^{\ln(1+h)^{1/h}}$$ لأن e ثابت

$$\therefore \quad e = \lim_{h \to 0} (1+h)^{1/h}$$ $e^{\ln[f(x)]} = f(x)$ قاعدة

or $$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}$$

Problems

- Differentiate the function.

(a) $f(x) = \log(1 + \cos x)$

(b) $g(x) = \ln(xe^{-2x})$

(c) $F(t) = (\ln t)^2 \sin t$

(d) $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

(e) $P(v) = \frac{\ln v}{1-v}$

(f) $y = \ln\left|1 + t - t^3\right|$

(g) $y = \ln(e^{-x} + xe^{-x})$

(h) $g(x) = \ln \frac{x+1}{\sqrt{x-2}}$

(i) $H(z) = \ln\sqrt{\frac{a^2-z^2}{a^2+z^2}}$

(j) $y = \tan[\ln(ax+b)]$

- Find y' and y''.

$$y = \frac{\ln x}{1 + \ln x}$$

- Differentiate f and find the domain of f.

$$f(x) = \frac{x}{1 - \ln(x - 1)}$$

- Use logarithmic differentiation to find the derivative of the function.

(a) $y = \sqrt{x}e^{x^2}(x^2 + 1)^{10}$

(b) $y = x^x$

(c) $y = x^{\sin x}$

(d) $y = \left(\sqrt{x}\right)^{x}$