Section 3.9 – Related Rates

- الهدف: استخدام الاشتقاق لحل معدل تغير كمية بمرور الزمن .

Example 1

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Solution

$$\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}, \quad D = 50 \text{ cm} \Rightarrow r = 25 \text{ cm}, \quad \frac{dr}{dt} = ?$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cancel{3} r^2 \frac{dr}{dt}$$

$$100 = 4\pi (25)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{25\pi} \text{ cm}/\text{s}$$

- خطوات الحل:

1. اقرأ المسألة بعناية
2. ارسم المسألة بناءً على الوصف (إذا أمكن)
3. اكتب الرمز المناسب لكل معطى
4. أي كمية متغيرة مع الزمن تكتب كمشتقة
5. اكتب معادلة لربط المعطيات. اكتب كمية بدلالة كمية أخرى إذا أمكن (يمكن الاستعانة برسم المسألة
6. استخدم chain rule لاشتقاق طرفي المعادلة بالنسبة ل t
7. عوض بالقيم في المعادلة الأخيرة وحل لإيجاد المطلوب.

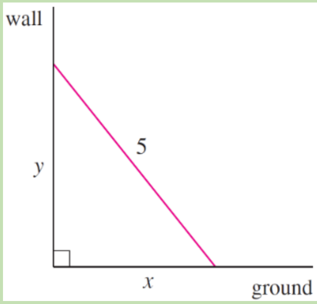
Example 2

A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 m from the wall?

Solution

$L = 5m, \ \frac{dx}{dt} = 1 \ m/s, \ x = 3m$

$\frac{dy}{dt} = ?$



$$x^2 + y^2 = 5^2$$

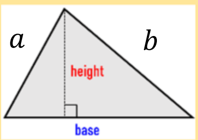
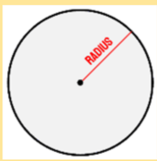
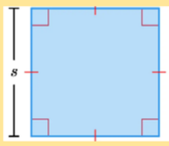
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

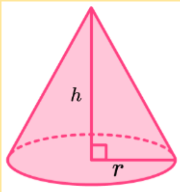
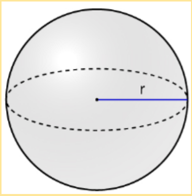
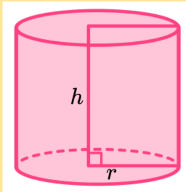
$$y = \sqrt{5^2 - x^2} = \sqrt{25 - 9} = 4$$

$$\therefore \quad 2 \cdot 3 \cdot 1 + 2 \cdot 4 \frac{dy}{dt} = 0$$

$$8\frac{dy}{dt} = -6 \Rightarrow \frac{dy}{dt} = \frac{-6}{8} = \frac{-3}{4} \ m/s$$

تذكر

	 Triangle	 Circle	 Rectangle	 Square
Area	$\frac{1}{2}bh$	πr^2	Lw	s^2
Circumference	$a+b+c$	$2\pi r$	$2L+2w$	$4s$

	 Cone	 Sphere	 Cylinder
Volume	$\frac{1}{3}\pi r^2 h$	$\frac{4}{3}\pi r^3$	$\pi r^2 h$
Surface area	$\pi r^2 + \pi r\sqrt{r^2 + h^2}$	$4\pi r^2$	$2\pi rh + 2\pi r^2$

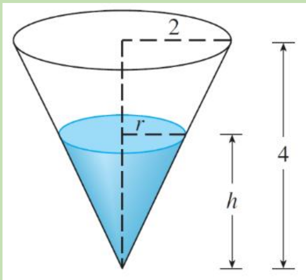
Example 3

A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

Solution

$r = 2\text{ m}, \quad h = 4\text{ m}, \quad \frac{dV}{dt} = 2\text{ m}^3/\text{min}$

$h = 3\text{ m}, \quad \frac{dh}{dt} = ?$



$$V = \frac{1}{3}\pi r^2 h$$

$$r = \frac{1}{2}h$$

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{\cancel{12}_{4}}\pi\,\cancel{3}h^2\frac{dh}{dt}$$

$$2 = \frac{1}{4}\pi\, 3^2\,\frac{dh}{dt}$$

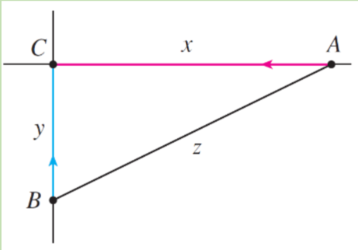
$$\frac{dh}{dt} = \frac{8}{9\pi}\ \text{m/min}$$

Example 4

Car A is traveling west at 90 km/h and car B is traveling north at 100 km/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 60 m and car B is 80 m from the intersection?

Solution

$$\frac{dx}{dt} = -90 \text{ km/h}, \quad \frac{dy}{dt} = -100 \text{ km/h}$$



$$x = 60 \text{ m} = 0.06 \text{ km}$$

$$y = 80 \text{ m} = 0.08 \text{ km}$$

$$\frac{dz}{dt} = ?$$

$$z^2 = x^2 + y^2$$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \qquad \div 2z$$

$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} + \frac{y}{z}\frac{dy}{dt}$$

$$z = \sqrt{(0.06)^2 + (0.08)^2}$$

$$= \sqrt{0.0036 + 0.0064}$$

$$= \sqrt{0.01} = 0.1 \text{ km}$$

$$\therefore \frac{dz}{dt} = \frac{0.06}{0.1} \cdot -90 + \frac{0.08}{0.1} \cdot -100$$

$$= -54 - 80 = -134 \text{ km/h}$$

Problems

- (a) If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt.

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

(b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?

$$\frac{dr}{dt} = 1 \text{ m/s}, \quad r = 30 \text{ m}, \quad \frac{dA}{dt} = ?$$

$$\frac{dA}{dt} = 2\pi \cdot 30 \cdot 1$$

$$= 60\pi \text{ m}^2/\text{s}$$

- The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

$$\frac{dL}{dt} = 8 \text{ cm/s}, \quad \frac{dw}{dt} = 3 \text{ cm/s}, \quad L = 20 \text{ cm}, \quad w = 10 \text{ cm}$$

$$\frac{dA}{dt} = ?$$

$$A = Lw$$

$$\frac{dA}{dt} = L \frac{dw}{dt} + \frac{dL}{dt} \cdot w$$

$$= 20 \cdot 3 + 8 \cdot 10 = 140 \text{ cm}^2/\text{s}$$

- The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?

$\frac{dr}{dt} = 4 \text{ mm/s}, \quad D = 80 \text{ mm} \Rightarrow r = 40 \text{ mm}, \quad \frac{dV}{dt} = ?$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cancel{3} r^2 \frac{dr}{dt}$

$= 4\pi (40)^2 \cdot 4 = 25{,}600\pi \quad \text{mm}^3/\text{s}$

- Suppose $4x^2 + 9y^2 = 36$, where x and y are functions of t.

(a) If $dy/dt = \frac{1}{3}$, find dx/dt when $x = 2$ and $y = \frac{2}{3}\sqrt{5}$.

$8x \frac{dx}{dt} + 18y \frac{dy}{dt} = 0$

$\frac{dx}{dt} = -\frac{18y}{8x} \frac{dy}{dt} = -\frac{\overset{6\,2}{\cancel{18}} \cdot \cancel{2}/\cancel{3}\sqrt{5}}{4\,\cancel{8} \cdot \cancel{2}} \cdot \frac{1}{\cancel{3}} = -\frac{\sqrt{5}}{4}$

(a) If $dx/dt = 3$, find dy/dt when $x = -2$ and $y = \frac{2}{3}\sqrt{5}$.

$\frac{dy}{dt} = \frac{-8x}{18y} \frac{dx}{dt}$

$= \frac{-\overset{4}{\cancel{8}} \cdot -\cancel{2}}{\underset{2}{\cancel{6}}\,\cancel{18} \cdot \cancel{2}/\cancel{3}\sqrt{5}} \cdot \cancel{3} = \frac{4}{\sqrt{5}}$

- If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

(a) What quantities are given in the problem?

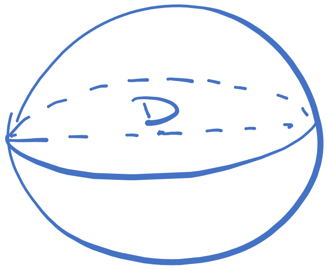
$$\frac{dA}{dt} = -1 \ cm^2/min$$

$$D = 10 \ cm$$

(b) What is the unknown?

$$\frac{dD}{dt}$$

(c) Draw a picture of the situation for any time t.



(d) Write an equation that relates the quantities.

$$A = 4\pi r^2 = 4\pi\left(\frac{D}{2}\right)^2 = \cancel{4}\pi \frac{D^2}{\cancel{4}} = \pi D^2$$

$$\frac{dA}{dt} = 2\pi D \frac{dD}{dt}$$

(e) Finish solving the problem.

$$-1 = 2\pi \cdot 10 \frac{dD}{dt}$$

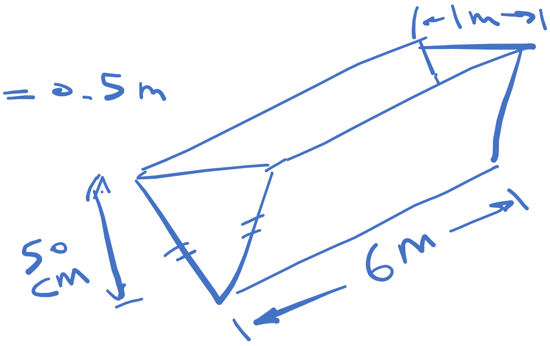
$$\frac{dD}{dt} = \frac{-1}{20\pi} \ cm/min$$

- A trough is 6 m long and its ends have the shape of isosceles triangles that are 1 m across at the top and have a height of 50 cm. If the trough is being filled with water at a rate of 1.2 m^3/min, how fast is the water level rising when the water is 30 cm deep?

$L = 6m, \; b = 1m, \; h = 50cm = 0.5m$

$\frac{dV}{dt} = 1.2 \; m^3/min, \; h = 30 cm = 0.3m$

$\frac{dh}{dt} = ?$



$V = \frac{1}{2} b h L$

$\because b = 2h$

$\therefore V = \frac{1}{2}(2h) h 6 = 6h^2$

$\frac{dV}{dt} = 12 h \frac{dh}{dt}$

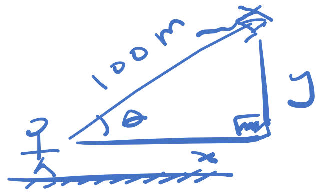
$1.2 = 12 \times 0.3 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1.2}{3.6} = \frac{1}{3} \; m/min$

- A kite 50 m above the ground moves horizontally at a speed of 2 m/s. At what rate is the angle between the string and the horizontal decreasing when 100 m of string has been let out?

$y = 50\text{m}, \quad \dfrac{dx}{dt} = 2\text{ m/s}$

$L = 100\text{m}, \quad \dfrac{d\theta}{dt} = ?$



$$\cos\theta = \frac{x}{100}$$

$$100\cos\theta = x$$

$$-100\sin\theta \cdot \frac{d\theta}{dt} = \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{dx/dt}{100\sin\theta}$$

$$\because \sin\theta = \frac{50}{100}$$

$$\therefore \frac{d\theta}{dt} = -\frac{2}{\cancel{100} \cdot \frac{50}{\cancel{100}}}$$

$$= -\frac{2}{50} = -\frac{1}{25} \text{ rad/s}$$