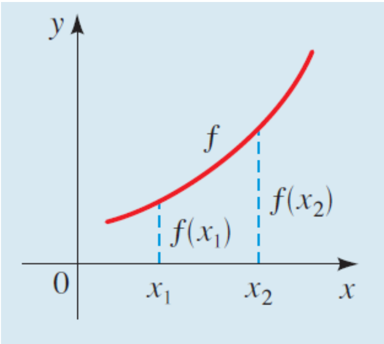
Section 4.1 – Maximum and Minimum Values

- Increasing function

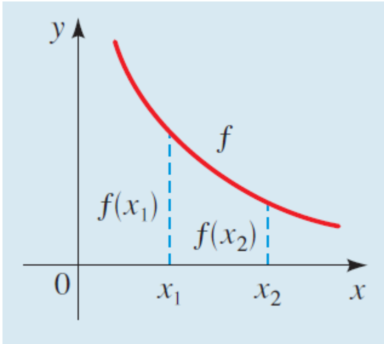
f is increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I

- Decreasing function

f is decreasing on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I



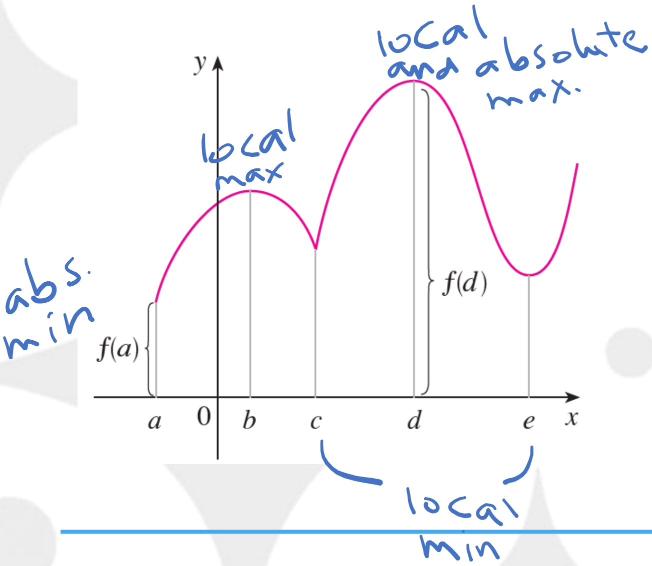
f is increasing

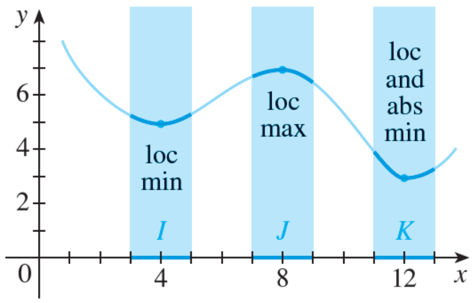


f is decreasing

- Extrema of a function القيم القصوى

Local		Absolute	
Maximum	Minimum	Maximum	Minimum
Let c be a number in the domain D of a function f. Then $f(c)$ is a • local maximum value of f if $f(c) \geq f(x)$ when x is near c.	Let c be a number in the domain D of a function f. Then $f(c)$ is a • local minimum value of f if $f(c) \leq f(x)$ when x is near c.	Let c be a number in the domain D of a function f. Then $f(c)$ is the • absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D.	Let c be a number in the domain D of a function f. Then $f(c)$ is the • absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.



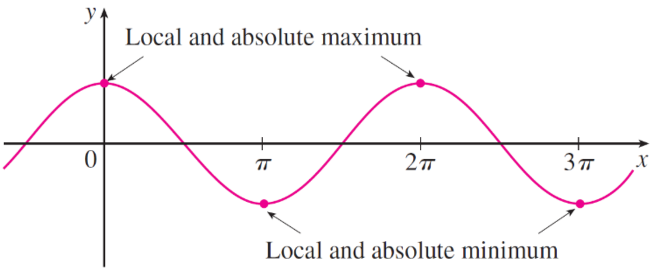


مثال

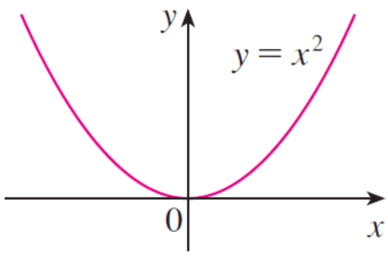
$f(x) = \cos x$ عندها عدد غير نهائي من maximum و minimum

لأن $\cos 2n\pi = 1$ و n تساوي أي عدد صحيح (integer)

و $\cos(2n+1)\pi = -1$



مثال

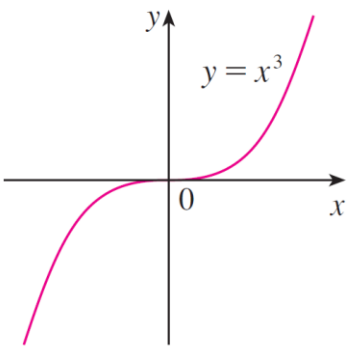


$f(x) = x^2$ لها minimum (local و absolute) عند $x=0$

لأن $f(x) \geq f(0)$ لكل قيم x

وليس لها max. لأنها تزيد إلى ما لا نهاية

مثال



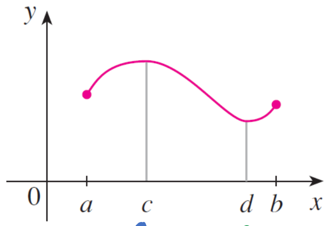
$f(x) = x^3$

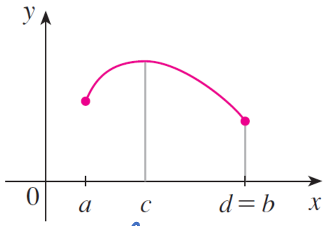
ليس لها أي extreme values (max. or min.)

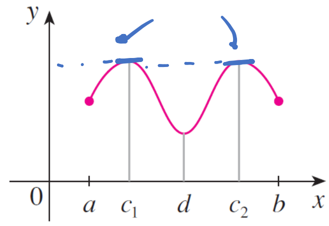
- **The Extreme Value Theorem**:

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

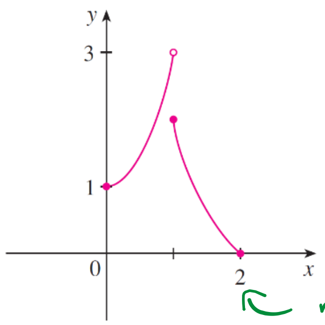
abs. max.





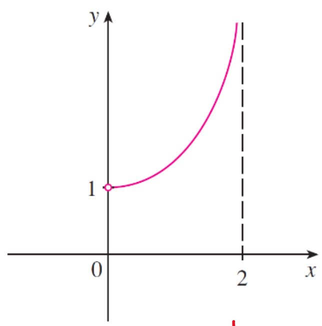


abs. max   abs. min   abs. max.   abs. min.   abs. min.



minimum

NO max.



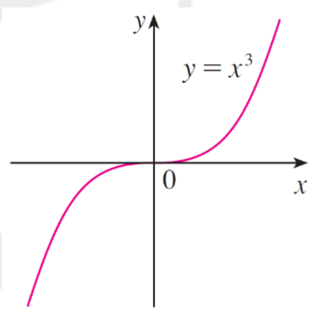
No extrema (max. or min.)

- **Fermat's Theorem:**

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

والعكس غير صحيح

مثال



$f(x) = x^3$

$f'(x) = 3x^2$

$f'(0) = 0$

لكن $x=0$ ليست max أو min

- A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 1

Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

Solution

$$f'(x) = x^{3/5}(-1) + 3/5\, x^{-2/5}(4-x)$$

$$= -x^{3/5} + \frac{3(4-x)}{5x^{2/5}}$$

$$= \frac{-5x + 3(4-x)}{5x^{2/5}}$$

$$= \frac{-5x + 12 - 3x}{5x^{2/5}}$$

$$= \frac{12 - 8x}{5x^{2/5}}$$

$f'(x) = 0 \Rightarrow 12 - 8x = 0 \Rightarrow x = \frac{-12}{-8} = \frac{3}{2}$

$f'(x)$ Does not exist for $x = 0$

$\therefore$ critical numbers are 0 and $\frac{3}{2}$

لاحظ

If f has a local maximum or minimum at c, then c is a critical number of f.

أي قيمة لـ x عندها local max. أو local min. هي critical number

- **The Closed interval Method**

لإيجاد absolute max. و absolute min. للدالة

متصلة في فترة مغلقة $[a,b]$

① نوجد critical numbers في الفترة، وقيم الدالة عندها

② نوجد قيم الدالة عند a و b (نهايتا الفترة)

③ أكبر قيمة للدالة من الخطوتين السابقتين هي abs. max.

وأصغر قيمة هي abs. min.

Example 2

Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \le x \le 4$$

Solution

$$f'(x) = 3x^2 - 6x$$

$$3x^2 - 6x = 0$$

$$3x(x-2) = 0$$

$$x = 0 \quad \text{or} \quad x = 2$$

$$f(0) = 1$$

$$f(2) = 2^3 - 3 \cdot 2^2 + 1 = -3$$

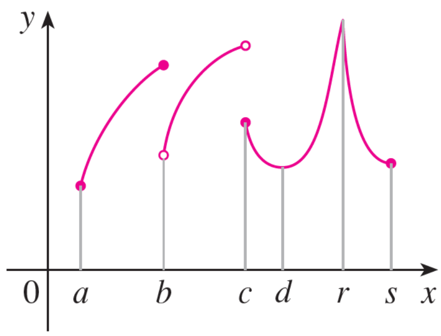
$$f\left(-\tfrac{1}{2}\right) = \left(-\tfrac{1}{2}\right)^3 - 3\left(-\tfrac{1}{2}\right)^2 + 1 = \frac{1}{8}$$

$$f(4) = 4^3 - 3 \cdot 4^2 + 1 = 17$$

$f(4) = 17$ is abs. max. , $f(2) = -3$ is abs. min

Problems

- For each of the numbers a, b, c, d, r, and s, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.



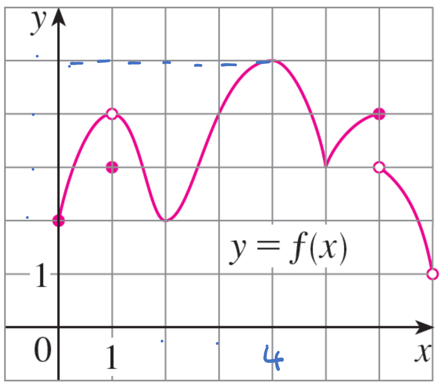
abs. max. at r

abs. min. at a

local min. at d

no extrema at b, c, or s

- Use the graph to state the absolute and local maximum and minimum values of the function.



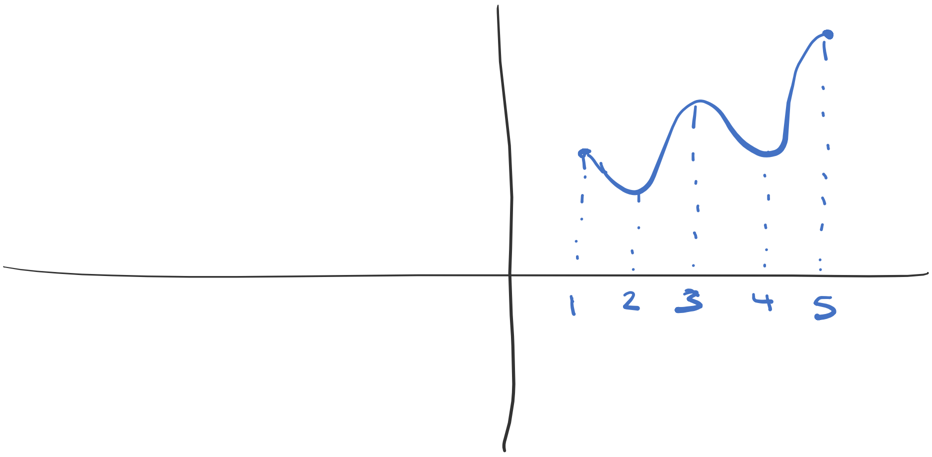
absolute maximum
at $x = 4$
$f(4) = 5$

No local max.

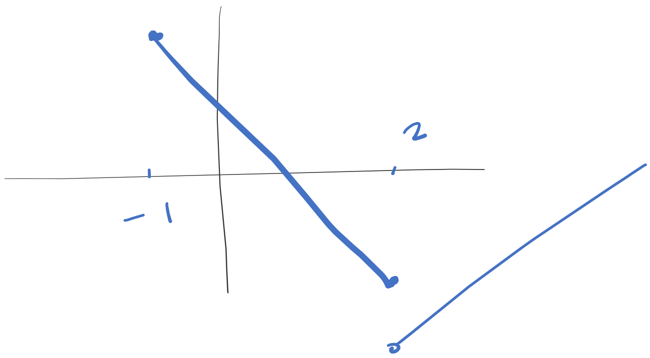
local minimum at $x = 2$, $x = 5$

No abs. min.

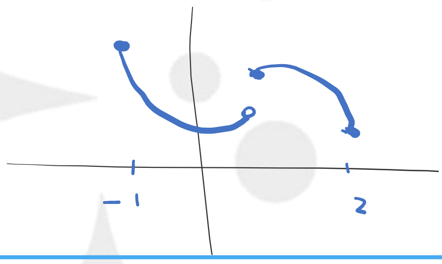
- Sketch the graph of a function f that is continuous on $[1, 5]$ and has the given properties: Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4



- (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no absolute minimum.



(b) Sketch the graph of a function on $[-1, 2]$ that is discontinuous but has both an absolute maximum and an absolute minimum.



- Find the critical numbers of the function.

(a) $g(x) = \sqrt[3]{4 - x^2}$

$$g'(x) = \frac{1}{3}(4 - x^2)^{-2/3} \cdot (-2x)$$

$$= \frac{-2x}{3(4 - x^2)^{2/3}}$$

$g'(x) = 0 \Rightarrow -2x = 0 \Rightarrow x = 0$

$g'(x)$ doesn't exist for $4 - x^2 = 0$

$x^2 = 4 \rightarrow x = 2$ or $x = -2$

$\therefore -2, 0, 2$ are critical numbers

(b) $f(\theta) = 2\cos\theta + \sin^2\theta$

$f'(\theta) = -2\sin\theta + 2\sin\theta \cdot \cos\theta$

$f'(\theta) = 0 \Rightarrow 2\sin\theta(\cos\theta - 1) = 0$

$\sin\theta = 0$ or $\cos\theta = 1$

$\theta = n\pi$ $\quad\quad$ $\theta = 2n\pi$

$\therefore$ critical number $n\pi$

($n\pi$ includes $2n\pi$)

(c) $f(x) = x^2 e^{-3x}$

$$f'(x) = x^2 \cdot -3e^{-3x} + 2xe^{-3x}$$

$$= -3x^2 e^{-3x} + 2xe^{-3x}$$

$$f'(x) = 0 \Rightarrow -3x^2 e^{-3x} + 2xe^{-3x} = 0$$

$$e^{-3x}(-3x^2 + 2x) = 0$$

$e^{-3x} \neq 0$

$$-3x^2 + 2x = 0$$

$$x(-3x + 2) = 0$$

$$x = 0 \quad \text{or} \quad x = \frac{2}{3}$$

critical numbers are 0 and $\frac{2}{3}$

- Find the absolute maximum and absolute minimum values of f on the given interval.

(a) $f(x) = 2x^3 - 3x^2 - 12x + 1, \qquad [-2, 3]$

$f'(x) = 6x^2 - 6x - 12$

$6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2 \quad \text{or} \quad x = -1$

$f(2) = 2 \cdot 2^3 - 3 \cdot 2^2 - 12 \cdot 2 + 1 = -19$

$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1 = 8$

$f(-2) = 2(-2)^3 - 3(-2)^2 - 12(-2) + 1 = -3$

$f(3) = 2(3)^3 - 3(3)^2 - 12(3) + 1 = -8$

$f(2) = -19$ is absolute max.

$f(-1) = 8$ is absolute min.

(b) $f(x) = x^{-2}\ln x, \qquad \left[\frac{1}{2}, 4\right]$

$$f'(x) = x^{-2}\cdot\frac{1}{x} - 2x^{-3}\ln x$$

$$= \frac{1}{x^3} - \frac{2\ln x}{x^3} = \frac{1-2\ln x}{x^3}$$

$$f'(x) = 0 \Rightarrow \quad 1 - 2\ln x = 0$$

$$\ln x^2 = 1$$

$$x^2 = e$$

$$x = \pm\sqrt{e}$$

$f'(x)$ doesn't exist for $x = 0$

$x = \sqrt{e}$ is the only critical point in the interval

$$f(\sqrt{e}) = \frac{\ln e^{1/2}}{(\sqrt{e})^2} = \frac{1/2}{e} = \frac{1}{2e}$$

$$f\left(\tfrac{1}{2}\right) = \frac{\ln 1/2}{(1/2)^2} = 4\ln \tfrac{1}{2} \quad < 0 \qquad \ln \tfrac{1}{2} < 0$$

$$f(4) = \frac{\ln 4}{4^2} = \frac{\ln 4}{16}$$

$f\left(\frac{1}{2}\right)$ is absolute min.

$f(\sqrt{e})$ is absolute max.

(c) $f(x) = xe^{x/2}, \qquad [-3, 1]$

$f'(x) = x \cdot \frac{1}{2} e^{x/2} + e^{x/2}$

$f'(x) = 0 \Rightarrow \quad \frac{1}{2} x e^{x/2} + e^{x/2} = 0$

$e^{x/2} \neq 0 \qquad e^{x/2}\left(\frac{1}{2}x + 1\right) = 0$

$\frac{1}{2}x + 1 = 0$

$x = -2$

$f(-2) = -2e^{-1} = \frac{-2}{e}$

$f(-3) = -3e^{-3/2} = \frac{-3}{e^{3/2}}$

$f(1) = e^{1/2}$

$f(1) = \sqrt{e}$ is abs. max

$f(-3) = \frac{-3}{e^{3/2}}$ is abs. min