Section 4.2 – The Mean Value Theorem

- The Rolle's Theorem: نحفظ

Let f be a function that satisfies the following three hypotheses:

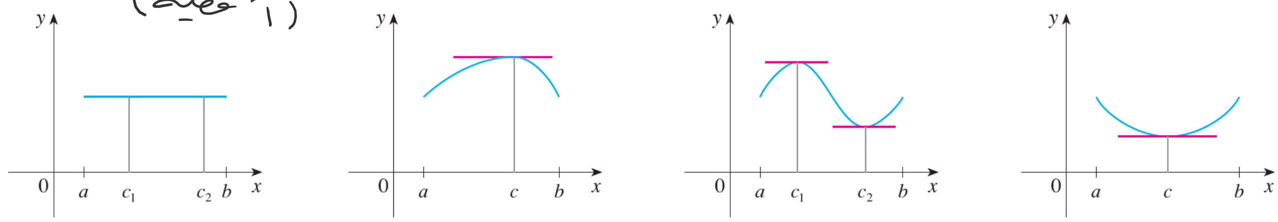
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

معنى $f(a) = f(b)$

الدالة ثابتة (أفقية)

الدالة صعدت ثم هبطت أو هبطت ثم صعدت



(a) (b) (c) (d)

Example 1

Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Solution

نبدأ بـ IVT

$f(x) = x^3 + x - 1$

polynomial $\Rightarrow$ continuous $\forall x \in \mathbb{R}$

$f(0) = -1 < 0$

$f(1) = 1 > 0$

there's a number $0 < c < 1$

such that $f(c) = 0$ (c is a real root)

If we have two (or more) roots then for the two roots a and b

$f(a) = f(b) = 0$

and since $f(x)$ is continuous and differentiable everywhere (poly)

then $\exists\, c \in (a,b)$ such that $f'(c) = 0$

by Rolle's Theorem

but $f'(x) = 3x^2 + 1 \geq 1 \quad \forall x$

therefore it's not possible for $f(x)$ to have more than one root by contradiction

تناقض

- **The Mean Value Theorem**:

Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

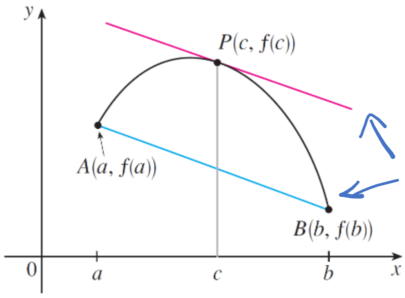
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

ميل الخط الواصل بين a و b

$$m_{AB} = \frac{f(b) - f(a)}{b - a}$$



متوازيين

Example 2

Use Mean Value Theorem to show that $\exists c \in [0,2]$ such that $f'(c) = 3$, where $f(x) = x^3 - x$.

Solution

$f(x)$ is continuous and differentiable on $(0,2)$ (polynomial)

then by MVT $\exists c \in [0,2]$. such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

$$= \frac{(2^3 - 2) - 0}{2}$$

$$= \frac{6}{2} = 3$$

Example 3

Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all values of x. How large can $f(2)$ possibly be?

Solution

Assume $f(x)$ is continuous on $[0,2]$
and differentiable on $(0,2)$

then by MVT $\exists\, c \in (0,2)$, such that

$$f(2) - f(0) = f'(c)(2-0)$$

$$f(2) - (-3) = 2f'(c)$$

$$f(2) + 3 \le 10$$

$$\therefore\ f(2) \le 7$$

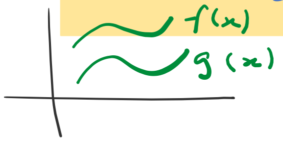
لاحظ

- If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

ارجع أول رسم في المذكرة

- If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

رسم الدالتين متماثلتين ، لكن واحدة أعلى من الاخرى



Example 4

Prove the identity $\tan^{-1} x + \cot^{-1} x = \pi/2$.

Solution

$$\left[\tan^{-1} x + \cot^{-1} x\right]' = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

and $\left(\pi/2\right)' = 0$

$$\therefore \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} + c$$

$$\tan^{-1}(1) + \cot^{-1}(1) = \frac{\pi}{2} + c$$

$$\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} + c$$

and hence $c = 0$

$$\therefore \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

Problems

- **Let $f'(x) = \frac{1}{3+2x^2}$, $\forall x \in \mathbb{R}$ and $f(1) = 0$. Show that $\frac{1}{11} < f(2) < \frac{1}{5}$**

$f'(x)$ exists for $\forall x \in \mathbb{R}$

therefore $f(x)$ is differentiable on $(1,2)$

and hence continuous on $[1,2]$

then by MVT $\exists\, c \in (1,2)$ such that

$$f(2) - \cancel{f(1)}^{\,0} = f'(c)(2-1)$$

$$f(2) = f'(c)$$

for $1 < c < 2$

$$3 + 2(1)^2 < 3 + 2c^2 < 3 + 2(2)^2$$

$$5 < 3 + 2c^2 < 11$$

$$\frac{1}{5} > \frac{1}{3+2c^2} > \frac{1}{11}$$

$$\therefore \quad \frac{1}{11} < f(2) < \frac{1}{5}$$

- **Use the Mean Value Theorem to show that**

$(3+x)^{\frac{1}{3}} < 2 + \frac{1}{2}(x-5), \forall x > 5$

$f(x) = (3+x)^{1/3}$ is continuous on $[5, x]$

and differentiable on $(5, x)$

then by MVT $\exists\, c \in (5, x)$ such that

$$f(x) - f(5) = f'(c)\,(x-5)$$

$$(3+x)^{1/3} - (3+5)^{1/3} = \tfrac{1}{3}(3+c)^{-2/3}\,(x-5)$$

$$(3+x)^{1/3} = 2 + \frac{1}{3}\,\frac{1}{(3+c)^{2/3}}\,(x-5)$$

$$\because\ \frac{1}{3}\,\frac{1}{(3+c)^{2/3}}(x-5) < \frac{1}{2}(x-5)$$

$$\therefore\ (3+x)^{1/3} < 2 + \tfrac{1}{2}(x-5)$$

- Let $f(x) = \frac{x+1}{x-1}$

(a) Show that there is no real number $c \in (0,2)$ such that $f(2) - f(0) = 2f'(c)$

$$f'(x) = \frac{(x-1) - (x+1)}{(x-1)^2}$$

$$= \frac{-2}{(x-1)^2}$$

$$f(2) - f(0) = 2f'(c)$$

$$\frac{2+1}{2-1} - \frac{0+1}{0-1} = 2 \cdot \frac{-2}{(c-1)^2}$$

$$4 = \frac{-4}{(c-1)^2} \Rightarrow 4(c-1)^2 = -4$$

$(c-1)^2 = -1$ No real solution

$\therefore$ there is no real number $c \in (0,2)$ such that $f(2) - f(0) = 2f'(c)$

(b) Why does this not contradict the Mean Value Theorem?

Because $f(x)$ doesn't exist at $x=1$ and $1 \in [0,2]$

- Let $f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 < x \le 2 \end{cases}$

(a) Show that $f(0) = f(2)$

$f(0) = 0$

$f(2) = 2 - 2 = 0$

$\therefore f(0) = f(2)$

(b) Show that $f'(c) \neq 0, \forall c \in (0, 2)$

$f'(x) = \begin{cases} 1 & 0 \le x \le 1 \\ -1 & 1 < x \le 2 \end{cases}$

$f'(c) \neq 0 \quad \forall c \in (0,2)$

(c) Does this contradict the Rolle's Theorem? Explain.

No. Because $f(x)$ is not differentiable at $x = 1$ ($f'(1)$ DNE) and $1 \in (0,2)$

- Suppose that f is a continuous function on $[a,b]$ and $f'(x) < 0, \forall x \in (a,b)$. Use the Mean Value Theorem to show that $f(b) < f(a)$

$f'(x)\ \forall x \in (a,b)$ exists (given)

$\therefore f(x)$ is differentiable on (a,b)

and hence continuous on $[a,b]$

then by MVT $\exists\, c \in (a,b)$ such that

$$f(b) - f(a) = \underbrace{f'(c)}_{-} \cdot \underbrace{(b-a)}_{+} < 0$$

$\therefore f(b) < f(a)$