

## Section 4.2 – The Mean Value Theorem

- The Rolle's Theorem:

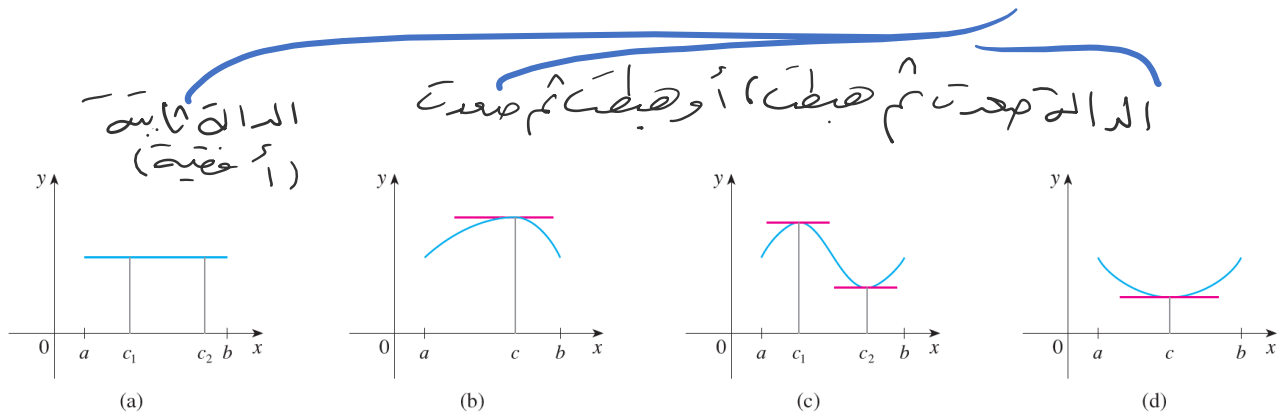
دالة

Let  $f$  be a function that satisfies the following three hypotheses:

1.  $f$  is continuous on the closed interval  $[a, b]$ .
2.  $f$  is differentiable on the open interval  $(a, b)$ .
3.  $f(a) = f(b)$

Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = 0$ .

معنى  $f(a) = f(b)$



### Example 1

Prove that the equation  $x^3 + x - 1 = 0$  has exactly one real root.

**Solution**

$$f(x) = x^3 + x - 1$$

polynomial  $\Rightarrow$  continuous  $\forall x \in \mathbb{R}$

$$f(0) = -1 < 0$$

$$f(1) = 1 > 0$$

there's a number  $0 < c < 1$

such that  $f(c) = 0$  ( $c$  is a real root)

ببدأ بـ INT

If we have two (or more) roots  
then for the two roots  $a$  and  $b$   
 $f(a) = f(b) = 0$

and since  $f(x)$  is continuous  
and differentiable everywhere (poly)

then  $\exists c \in (a, b)$  such that  $f'(c) = 0$   
by Rolle's Theorem

but  $f'(x) = 3x^2 + 1 \geq 1 \quad \forall x$

therefore it's not possible for  $f(x)$

to have more than one root by **contradiction**

ناقض

معنى  
- The Mean Value Theorem:

Let  $f$  be a function that satisfies the following hypotheses:

1.  $f$  is continuous on the closed interval  $[a, b]$ .
2.  $f$  is differentiable on the open interval  $(a, b)$ .

Then there is a number  $c$  in  $(a, b)$  such that

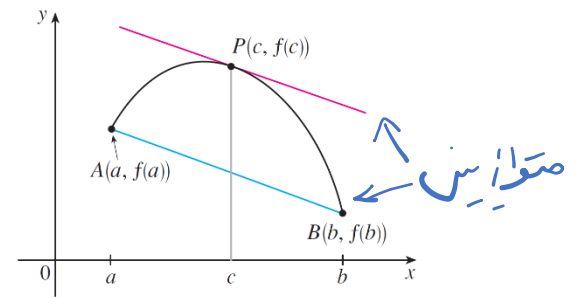
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

ميل الخط الواسل بين  $a$  و  $b$

$$m_{AB} = \frac{f(b) - f(a)}{b - a}$$



### Example 2

Use Mean Value Theorem to show that  $\exists c \in [0, 2]$  such that  $f'(c) = 3$ , where  $f(x) = x^3 - x$ .

#### Solution

$f(x)$  is continuous and differentiable on  $(0, 2)$  (polynomial)

then by MVT  $\exists c \in [0, 2]$  such that

$$\begin{aligned} f'(c) &= \frac{f(2) - f(0)}{2 - 0} \\ &= \frac{(2^3 - 2) - 0}{2} \\ &= \frac{6}{2} = 3 \end{aligned}$$

**Example 3**

Suppose that  $f(0) = -3$  and  $f'(x) \leq 5$  for all values of  $x$ . How large can  $f(2)$  possibly be?

**Solution**

Assume  $f(x)$  is continuous on  $[0, 2]$   
and differentiable on  $(0, 2)$   
then by MVT  $\exists c \in (0, 2)$ , such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

$$f(2) - (-3) = 2f'(c)$$

$$f(2) + 3 \leq 10$$

$$\therefore f(2) \leq 7$$

### لاحظ

- If  $f'(x) = 0$  for all  $x$  in an interval  $(a, b)$ , then  $f$  is constant on  $(a, b)$ .

اجمع أول رسم في المذكرة

- If  $f'(x) = g'(x)$  for all  $x$  in an interval  $(a, b)$ , then  $f - g$  is constant on  $(a, b)$ ; that is,  $f(x) = g(x) + c$  where  $c$  is a constant.

رسم الدالتين متماثلتين ، لكن واحدة أعلى من الأخرى



### Example 4

Prove the identity  $\tan^{-1} x + \cot^{-1} x = \pi/2$ .

#### Solution

$$\left[ \tan^{-1} x + \cot^{-1} x \right]' = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

$$\text{and } \left( \frac{\pi}{2} \right)' = 0$$

$$\therefore \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} + c$$

$$\tan^{-1}(1) + \cot^{-1}(1) = \frac{\pi}{2} + c$$

$$\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} + c$$

and hence  $c = 0$

$$\therefore \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

**Problems**

- Let  $f'(x) = \frac{1}{3+2x^2}$ ,  $\forall x \in \mathbb{R}$  and  $f(1) = 0$ . Show that  $\frac{1}{11} < f(2) < \frac{1}{5}$

- Use the Mean Value Theorem to show that

$$(3 + x)^{\frac{1}{3}} < 2 + \frac{1}{2}(x - 5), \forall x > 5$$

- Let  $f(x) = \frac{x+1}{x-1}$

(a) Show that there is no real number  $c \in (0, 2)$  such that  $f(2) - f(0) = 2f'(c)$

(b) Why does this not contradict the Mean Value Theorem?

- Let  $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 < x \leq 2 \end{cases}$

(a) Show that  $f(0) = f(2)$

(b) Show that  $f'(c) \neq 0, \forall c \in (0, 2)$

(c) Does this contradict the Rolle's Theorem? Explain.

- Suppose that  $f$  is a continuous function on  $[a, b]$  and  $f'(x) < 0, \forall x \in (a, b)$ .  
Use the Mean Value Theorem to show that  $f(b) < f(a)$