Section 4.3 – How Derivatives Affect the Shape of a Graph

- Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

إذا كانت المشتقة موجبة يعني الدالة بتزيد (صاعدة)

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

إذا كانت المشتقة سالبة يعني الدالة تقل (نازلة)

Example 1

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Solution

لإيجاد الحل، نحسب sign chart بالخطوات التالية

$$f'(x) = 12x^3 - 12x^2 - 24x \rightarrow 1$$

Since $f'(x)$ is polynomial, it exists $\forall x \in \mathbb{R}$

therefore critical points only exist for $f'(x) = 0$

$$12x^3 - 12x^2 - 24x = 0$$

$$12x(x^2 - x - 2) = 0$$

$$12x(x-2)(x+1) = 0$$

$$x = 0 \quad \text{or} \quad x = 2 \quad \text{or} \quad x = -1 \rightarrow 2$$

	$-\infty$ to -1	-1 to 0	0 to 2	2 to ∞
test value	-2	-0.5	1	3
$12x$	$-$	$-$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$	$+$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	↘	↗	↘	↗

(3)

$f(x)$ is increasing on $(-1,0) \cup (2,\infty)$
and is decreasing on $(-\infty,-1) \cup (0,2)$ (4)

- Suppose that c is a critical number of a continuous function f.

تغير إشارة f' حول نقطة معناه القمة max. أو min.

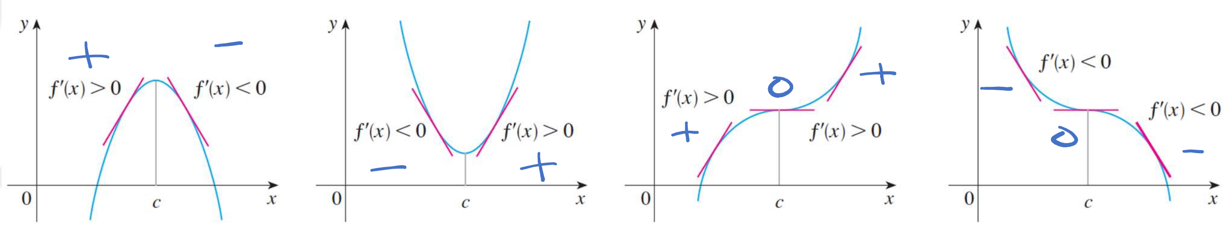
(a) If f' changes from positive to negative at c, then f has a local maximum at c.

$+ \rightarrow -$

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

$- \rightarrow +$

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.



(a) Local maximum (b) Local minimum (c) No maximum or minimum (d) No maximum or minimum

Example 2

Find the local minimum and maximum values of the function f in Example 1.

Solution

x		-1		0		2	
$f'(x)$	$-$		$+$		$-$		$+$
$f(x)$	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

$$f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 5 = 0$$

local min.

$$f(0) = 5$$ is a local max.

$$f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 + 5 = -27$$

local min.

Example 3

Find the local maximum and minimum values of the function

$$g(x) = x + 2\sin x \qquad 0 \le x \le 2\pi$$

Solution

$g'(x) = 1 + 2\cos x$

$g'(x)$ exist $\forall x \in \mathbb{R}$

$1 + 2\cos x = 0$

$\cos x = -\frac{1}{2}$

$x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$

	0 to $2\pi/3$	$2\pi/3$ to $4\pi/3$	$4\pi/3$ to 2π
$1+2\cos x$	$+$	$-$	$+$
$f(x)$	↗	↘	↗

$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + 2\sin\left(\frac{2\pi}{3}\right)$

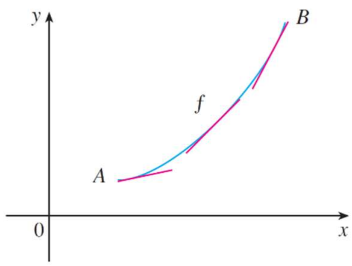
$= \frac{2\pi}{3} + \cancel{2}\,\frac{\sqrt{3}}{\cancel{2}} = \frac{2\pi}{3} + \sqrt{3}$ is local max.

$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + 2\sin\left(\frac{4\pi}{3}\right)$

$= \frac{4\pi}{3} + \cancel{2} \cdot \frac{-\sqrt{3}}{\cancel{2}} = \frac{4\pi}{3} - \sqrt{3}$ is local min.

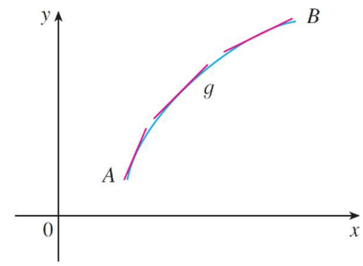
- Concavity:

If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.



Concave upward

لأن المنحنى فوق المماس

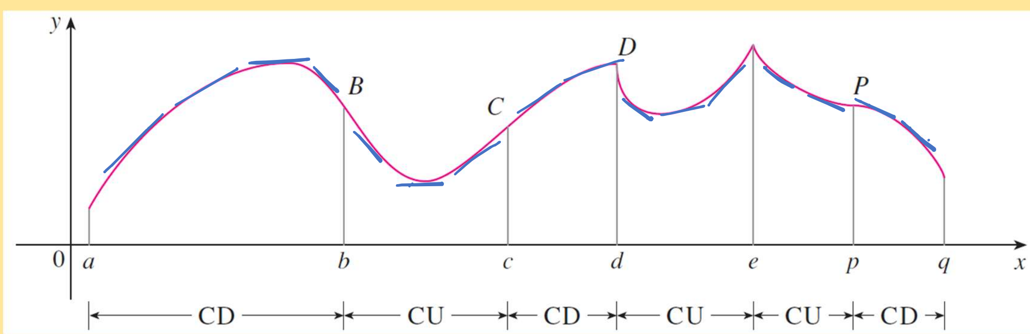


Concave downward

لأن المنحنى تحت المماس

لاحظ

ال concavity ليس لها علاقة بالصعود والهبوط



<u>Concavity Test</u> طريقة إيجاد ال concavity

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

f'' موجبة

f'' سالبة

Example 4

Discuss the curve $y = x^4 - 4x^3$ with respect to concavity.

Solution

$y' = 4x^3 - 12x^2$

$y'' = 12x^2 - 24x \rightarrow$ ①

$12x^2 - 24x = 0$

$12x(x-2) = 0$

$x = 0$ or $x = 2 \rightarrow$ ②

	$-\infty$ to 0 (test -1)	0 to 2 (test 1)	2 to ∞ (test 3)
$12x$	$-$	$+$	$+$
$x-2$	$-$	$-$	$+$
$f''(x)$	$+$	$-$	$+$
$f(x)$	$\cup$	$\cap$	$\cup$

$f(x)$ is concave up on $(-\infty, 0) \cup (2, \infty)$

and is concave down on $(0, 2)$

<u>Inflection point</u> نقطة تغير الـ concavity تسمى

A point P on a curve $y = f(x)$ is called an <u>inflection point</u> if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

- The Second Derivative Test: طريقة مختلفة لإيجاد local max/min

Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example 5

Discuss the points of inflection and local maxima and minima for the function in example 4.

Solution

		0		2	
$f''(x)$	$+$		$-$		$+$
$f(x)$	$\cup$		$\cap$		$\cup$

$x = 0, 2$ are inflection points

$f'(x) = 4x^3 - 12x^2$

$4x^3 - 12x^2 = 0$

$4x^2(x-3) = 0$

$x = 0$ or $x = 3$

$f''(0) = 0$ inconclusive $\Rightarrow$ نستخدم first derivative test

$f''(3) = 12(3)^2 - 24(3) = 36 > 0$

$f(3) = 3^4 - 4(3)^3 = -27$

point $(3, -27)$ is local min.

by the Second Derivative Test

لاحظ

The **Second Derivative Test** is <u>inconclusive</u> when $f''(c) = 0$.

يعني لا نستطيع الحكم !

لذلك في الاختبار، دائماً نستخدم 1st derivative test

إذا لم يحدد

Example 6

Discuss the curve $f(x) = x^{2/3}(6-x)^{1/3}$ with respect to concavity, points of inflection, and local maxima and minima.

Solution

$$f'(x) = x^{2/3} \cdot \frac{1}{3}(6-x)^{-2/3} \cdot -1 + \frac{2}{3} x^{-1/3} (6-x)^{1/3}$$

$$= \frac{-x^{2/3}}{3(6-x)^{2/3}} + \frac{2(6-x)^{1/3}}{3x^{1/3}}$$

$$= \frac{-x^{2/3} \cdot x^{1/3} + 2(6-x)^{2/3}(6-x)^{1/3}}{3x^{1/3}(6-x)^{2/3}}$$

$$= \frac{-x + 2(6-x)}{3x^{1/3}(6-x)^{2/3}}$$

$$= \frac{-x + 12 - 2x}{3x^{1/3}(6-x)^{2/3}} = \frac{\cancel{3}(4-x)}{\cancel{3}x^{1/3}(6-x)^{2/3}}$$

$$= \frac{4-x}{x^{1/3}(6-x)^{2/3}} \longrightarrow 1$$

$4 - x = 0 \Rightarrow x = 4$

$f'(x)$ DNE for $x = 0$ and $x = 6$

therefore critical points are $0, 4, 6$

	$-\infty$ to 0 (test -1)	0 to 4 (test 1)	4 to 6 (test 5)	6 to ∞ (test 7)
$4-x$	$+$	$+$	$-$	$-$
$x^{1/3}$	$-$	$+$	$+$	$+$
$(6-x)^{2/3}$	$+$	$+$	$+$	$+$
$f'(x)$	$-$	$+$	$-$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$

$f(0) = 0$ is a local min

$f(4) = 4^{2/3}(6-4)^{1/3} = 2^{5/3}$ is a local max.

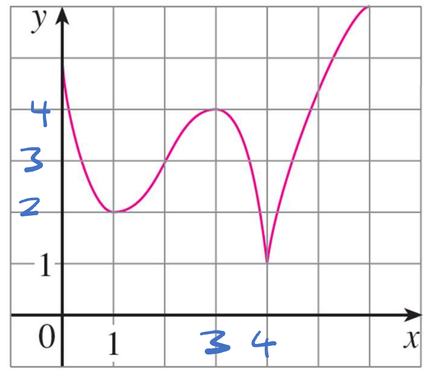
$$f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}$$

	$-\infty$ to 0 (test -1)	0 to 6 (test 1)	6 to ∞ (test 7)
-8	$-$	$-$	$-$
$x^{4/3}$	$+$	$+$	$+$
$(6-x)^{5/3}$	$+$	$+$	$-$
	$-$	$-$	$+$
	$\cap$	$\cap$	$\cup$

$f(6) = 0$ is the only inflection point

Problems

- Use the given graph of f to find the following.



(a) The open intervals on which f is increasing.

$(1,3) \cup (4,6)$

(b) The open intervals on which f is decreasing.

$(0,1) \cup (3,4)$

(c) The open intervals on which f is concave upward.

$(0,2)$

(d) The open intervals on which f is concave downward.

$(2,6)$

(e) The coordinates of the points of inflection.

$(2,3)$ is the only point of inflection

- For each of the following functions:

1. Find the intervals on which f is increasing or decreasing.

2. Find the local maximum and minimum values of f.

3. Find the intervals of concavity and the inflection points.

(a) $f(x) = \frac{x^2}{x^2+3}$

$$f'(x) = \frac{(x^2+3).2x - x^2(2x)}{(x^2+3)^2}$$

$$= \frac{\cancel{2x^3} + 6x - \cancel{2x^3}}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}$$

$f'(x)$ is defined everywhere

$6x = 0 \Rightarrow x = 0$ is the only cp

	$-\infty$	-1	0	1	∞
$6x$		$-$		$+$	
$(x^2+3)^2$		$+$		$+$	
$f'(x)$		$-$		$+$	
$f(x)$		$\searrow$		$\nearrow$	

$f(x)$ is increasing on $(-\infty, 0)$

and is decreasing on $(0, \infty)$

$f(0) = 0$ is local min.

$$f''(x) = \frac{(x^2+3)^2 \cdot 6 - 6x \cdot 2(x^2+3) \cdot 2x}{(x^2+3)^4}$$

$$= \frac{\cancel{(x^2+3)}\left[6 - 24x^2\right]}{(x^2+3)^{\cancel{4}\,3}} = \frac{6-24x^2}{(x^2+3)^3}$$

$$6 - 24x^2 = 0$$

$$6(1 - 4x^2) = 0$$

$$6(1 - 2x)(1 + 2x) = 0$$

$1 - 2x = 0$ or $1 + 2x = 0$

$x = \frac{1}{2}$ $\qquad$ $x = -\frac{1}{2}$

	$-\infty$ $\quad -1$ $\quad -\frac{1}{2}$	0 $\quad \frac{1}{2}$	1 $\quad \infty$
$6 - 24x^2$	$-$	$+$	$-$
$(x^2+3)^2$	$+$	$+$	$+$
$f''(x)$	$-$	$+$	$-$
$f(x)$	$\cap$	$\cup$	$\cap$

$f(x)$ concave up $\left(-\frac{1}{2}, \frac{1}{2}\right)$

concave down $\left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

$$f\left(\tfrac{1}{2}\right) = \frac{(1/2)^2}{(1/2)^2 + 3} = \frac{1}{4\left(\frac{1}{4} + 3\right)} = \frac{1}{13}$$

$$f\left(-\tfrac{1}{2}\right) = \frac{1}{13}$$

$\left(\frac{1}{2}, \frac{1}{13}\right)$ and $\left(-\frac{1}{2}, \frac{1}{13}\right)$ are inflection points

(b) $f(x) = \cos^2 x - 2\sin x, \quad 0 \le x \le 2\pi$

$$f'(x) = 2\cos x(-\sin x) - 2\cos x$$

$$-2\sin x\cos x - 2\cos x = 0$$

$$-2\cos x(\sin x + 1) = 0$$

$\cos x = 0$ or $\sin x = -1$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad$ $x = \frac{3\pi}{2}$

	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$-2\cos x$		$-$		$+$		$-$	
$\sin x + 1$		$+$		$+$		$+$	
$f'(x)$		$-$		$+$		$-$	
$f(x)$		$\searrow$		$\nearrow$		$\searrow$	

$f(x)$ is increasing $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

decreasing $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

$$f\left(\frac{\pi}{2}\right) = \overset{0}{\cancel{\left(\cos\frac{\pi}{2}\right)^2}} - 2\sin\frac{\pi}{2} = -2$$

$$f\left(\frac{3\pi}{2}\right) = \overset{0}{\cancel{\left(\cos\frac{3\pi}{2}\right)^2}} - 2\sin\frac{3\pi}{2} = 2$$

$\left(\frac{\pi}{2}, -2\right)$ is local min.

$\left(\frac{3\pi}{2}, 2\right)$ is local max.

$$f''(x) = -2[\sin x \cdot -\sin x + \cos x \cdot \cos x] + 2\sin x$$

$$= 2\sin^2 x - 2\cos^2 x + 2\sin x$$

$$= 2\sin^2 x - 2(1 - \sin^2 x) + 2\sin x$$

$$= 4\sin^2 x + 2\sin x - 2$$

$$4\sin^2 x + 2\sin x - 2 = 0 \qquad \div 2$$

$$2\sin^2 x + \sin x - 1 = 0$$

$$(2\sin x - 1)(\sin x + 1) = 0$$

$\sin x = \frac{1}{2}$ or $\sin x = -1$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ $\qquad$ $x = \frac{3\pi}{2}$

	0 to $\pi/6$	$\pi/6$ to $5\pi/6$ ($\frac{\pi}{4}$)	$5\pi/6$ to $3\pi/2$ (π)	$3\pi/2$ to 2π ($\frac{7\pi}{4}$)
$2\sin x - 1$	$-$	$+$	$-$	$-$
$\sin x + 1$	$+$	$+$	$+$	$+$
$f''(x)$	$-$	$+$	$-$	$-$
$f(x)$	$\cap$	$\cup$	$\cap$	$\cap$

$f(x)$ is concave up $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

concave down $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$

$f\left(\frac{\pi}{6}\right) = \left(\cos\frac{\pi}{6}\right)^2 - 2\sin\frac{\pi}{6} = -\frac{1}{4}$

$f\left(\frac{5\pi}{6}\right) = \left(\cos\frac{5\pi}{6}\right)^2 - 2\sin\frac{5\pi}{6} = -\frac{1}{4}$

are inflection points

(c) $f(x) = x^2 \ln x$

$$f'(x) = x^{\cancel{2}} \cdot \frac{1}{\cancel{x}} + 2x \ln x = x + 2x \ln x$$

$$f''(x) = 1 + 2\cancel{x} \cdot \frac{1}{\cancel{x}} + 2\ln x = 3 + 2\ln x$$

(d) $f(x) = \ln(x^2 + 9)$

$$f'(x) = \frac{2x}{x^2+9}$$

$$f''(x) = \frac{(x^2+9) \cdot 2 - 2x(2x)}{(x^2+9)^2}$$

$$= \frac{2x^2 + 18 - 4x^2}{(x^2+9)^2} = \frac{-2x^2+18}{(x^2+9)^2}$$

(e) $f(x) = \sqrt{x^2+1} - x$

$$f'(x) = \tfrac{1}{\cancel{2}}(x^2+1)^{-1/2} \cdot \cancel{2}x - 1$$

$$= \frac{x}{(x^2+1)^{1/2}} - 1$$

$$f''(x) = \frac{(x^2+1)^{1/2} - x \cdot \tfrac{1}{\cancel{2}}(x^2+1)^{-1/2} \cdot \cancel{2}x}{x^2+1}$$

$$= \frac{(x^2+1)^{1/2} - \dfrac{x^2}{(x^2+1)^{1/2}}}{x^2+1} \cdot \frac{(x^2+1)^{1/2}}{(x^2+1)^{1/2}}$$

$$= \frac{\cancel{x^2}+1 - \cancel{x^2}}{(x^2+1)^{3/2}} = \frac{1}{(x^2+1)^{3/2}}$$

(f) $f(x) = \ln(1 - \ln x)$

$$f'(x) = \frac{-1/x}{1-\ln x} = \frac{-1}{x - x\ln x}$$

$$f''(x) = \frac{0 - (-1)[1 - \cancel{x} \cdot \tfrac{1}{\cancel{x}} + \ln x]}{(x - x\ln x)^2}$$

$$= \frac{\ln x}{(x - x\ln x)^2}$$

- Find the local maximum and minimum values of f using both the First and Second Derivative Tests.

(a) $f(x) = x^5 - 5x + 3$

$$f' = 5x^4 - 5$$

$$5x^4 - 5 = 0 \Rightarrow 5x^4 = 5 \Rightarrow x^4 = 1$$

$$x = \pm 1$$

	$-\infty \quad -2 \quad -1$	$-1 \quad 0 \quad 1$	$1 \quad 2 \quad \infty$
$f'(x) = 5x^4 - 5$	$+$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

$$f(-1) = (-1)^5 - 5(-1) + 3 = 7$$

$$f(1) = 1^5 - 5(1) + 3 = -1$$

$(-1, 7)$ is a local max.
$(1, -1)$ is a local min. } by 1^{st} derivative test

$$f''(x) = 20x^3$$

$f''(-1) = 20(-1)^3 = -20 < 0$ local max.

$f''(1) = 20(1)^3 = 20 > 0$ local min.

by 2^{nd} derivative test

- (a) Find the critical numbers of $f(x) = x^4(x-1)^3$.

$$f'(x) = x^4 \cdot 3(x-1)^2 + 4x^3(x-1)^3$$

$$= x^3(x-1)^2[3x + 4(x-1)]$$

$$= x^3(x-1)^2(7x-4)$$

$$x^3(x-1)^2(7x-4) = 0$$

$$x = 0 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = \frac{4}{7}$$

$f'(x)$ is a polynomial $\Rightarrow$ defined $\forall x \in \mathbb{R}$

therefore $0, 1, \frac{4}{7}$ are the only critical points

(b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?

$$f''(x) = x^3(x-1)^2 7 + x^3 \cdot 2(x-1)(7x-4) + 3x^2(x-1)^2(7x-4)$$

$$= x^2(x-1)[7x(x-1) + 2x(7x-4) + 3(x-1)(7x-4)]$$

$$\left.\begin{aligned} f''(0) &= 0 \\ f''(1) &= 0 \end{aligned}\right\} \text{The Second Derivative Test is } \underline{\text{inconclusive}}$$

$$f''(4/7) = \left(\frac{4}{7}\right)^2\left(-\frac{3}{7}\right)\left[4\left(-\frac{3}{7}\right) + \cancel{\frac{8}{7}(4-4)} + \cancel{3\left(\frac{-3}{7}\right)(4-4)}\right]$$

$$= \left(\frac{4}{7}\right)^2\left(-\frac{3}{7}\right)4\left(-\frac{3}{7}\right) > 0$$

$\left(\frac{4}{7}, f\left(\frac{4}{7}\right)\right)$ is a local min

(c) What does the First Derivative Test tell you?

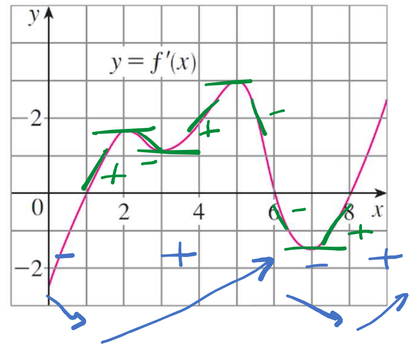
	$-\infty$ $\quad -1 \quad$ 0	0 $\quad \frac{1}{2} \quad$ $4/7$	$4/7$ $\quad \frac{3}{4} \quad$ 1	1 $\quad 2 \quad$ ∞
x^3	$-$	$+$	$+$	$+$
$(x-1)^2$	$+$	$+$	$+$	$+$
$(7x-4)$	$-$	$-$	$+$	$+$
$f'(x)$	$+$	$-$	$+$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$	$\nearrow$

$f(0) = 0$ is a local max.

$f\left(\frac{4}{7}\right) = \left(\frac{4}{7}\right)^4\left(\frac{4}{7} - 1\right)^3$ is a local min.

No local max or min at $x = 1$

- The graph of the derivative f' of a continuous function f is shown.



(a) On what intervals is f increasing? Decreasing?

Increasing

$(1,6) \cup (8,9)$

Decreasing

$(0,1) \cup (6,8)$

(b) At what values of x does f have a local maximum? Local minimum?

$f(1)$ and $f(8)$ are local minima

$f(6)$ is a local maximum

(c) On what intervals is f concave upward? Concave downward?

Concave up

$(0,2) \cup (3,5) \cup (7,9)$

Concave down

$(2,3) \cup (5,7)$

(d) State the x-coordinate(s) of the point(s) of inflection.

$x = 2, 3, 5$, and 7