Section 4.5 – Summary of Curve Sketching

- Guidelines for Sketching a Curve خطوات رسم دالة

1. Find the domain

- جذر زوجي : ماتحت الجذر موجب ≥ 0
- كسر : المقام $\neq 0$
- ln , log : مابداخلها > 0

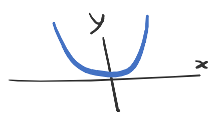
2. Intercepts التقاطعات

i. x-intercepts $\Rightarrow y = 0 \Rightarrow (x, 0)$

ii. y-intercepts $\Rightarrow x = 0 \Rightarrow (0, f(0))$

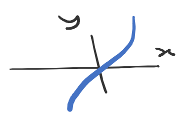
3. Symmetry التماثل

i. y-axis



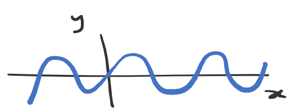
$f(-x) = f(x)$ Even function

ii. Origin



$f(-x) = -f(x)$ Odd function

iii. Periodicity



$f(x+k) = f(x)$

4. Asymptotes

i. Horizontal $\lim_{x \to \pm\infty} f(x) = a \Rightarrow y = a$

ii. Vertical $\lim_{x \to b} f(x) = \pm\infty \Rightarrow x = b$

5. Critical points $f'(x) = 0$

6. Intervals of Increase or Decrease, Local Maximum and Minimum Values

$f'(x) \begin{cases} + & \text{increase} \\ - & \text{decrease} \end{cases}$

7. Concavity and Points of Inflection

$f'' \begin{cases} + & \cup \\ - & \cap \end{cases}$

8. Sketch

Example 1

Use the guidelines to sketch the curve $y = \frac{2x^2}{x^2-1}$.

Solution

1. $D = \mathbb{R} \setminus \{-1, 1\}$

2. $2x^2 = 0 \Rightarrow x = 0$

$(0,0)$ is the only intercept

3. $f(-x) = \frac{2(-x)^2}{(-x)^2-1} = \frac{2x^2}{x^2-1} = f(x)$

$\therefore f(x)$ is even $\Rightarrow$ symmetric about y-axis

4. $\lim_{x \to \infty} \frac{2x^2}{x^2-1} \cdot \frac{\div x^2}{\div x^2} = \lim_{x \to \infty} \frac{2}{1 - 1/x^2} = 2$

$\lim_{x \to -\infty} \frac{2x^2}{x^2-1} \cdot \frac{\div x^2}{\div x^2} = \lim_{x \to \infty} \frac{2}{1 - 1/x^2} = 2$

$y = 2$ is HA

$x^2 - 1 = 0 \Rightarrow x = \pm 1$

$\lim_{x \to 1^-} \frac{2x^2}{x^2-1} = \frac{2}{0^-} = -\infty$ | $\lim_{x \to -1^-} \frac{2x^2}{x^2-1} = \frac{2}{0^+} = \infty$

$\lim_{x \to 1^+} \frac{2x^2}{x^2-1} = \frac{2}{0^+} = \infty$ | $\lim_{x \to -1^+} \frac{2x^2}{x^2-1} = \frac{2}{0^-} = -\infty$

$x = 1$ and $x = -1$ are VA

5. $$f'(x) = \frac{(x^2-1)\cdot 4x - 2x^2(2x)}{(x^2-1)^2}$$

$$= \frac{\cancel{4x^3} - 4x - \cancel{4x^3}}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

$f'(x) = 0 \Rightarrow -4x = 0 \Rightarrow x = 0$

$f'(x)$ DNE $\Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$

$0, -1, 1$ are critical points

6.

	$-\infty$ to -1 (test -2)	-1 to 0 (test $-\frac{1}{2}$)	0 to 1 (test $\frac{1}{2}$)	1 to ∞ (test 2)
$-4x$	+	+	−	−
$(x^2-1)^2$	+	+	+	+
$f'(x)$	+	+	−	−
$f(x)$	↗	↗	↘	↘

$f(0) = 0$ is a local max.

7. $$f''(x) = \frac{(x^2-1)^2 \cdot -4 - (-4x)\,2(x^2-1)\cdot 2x}{(x^2-1)^4}$$

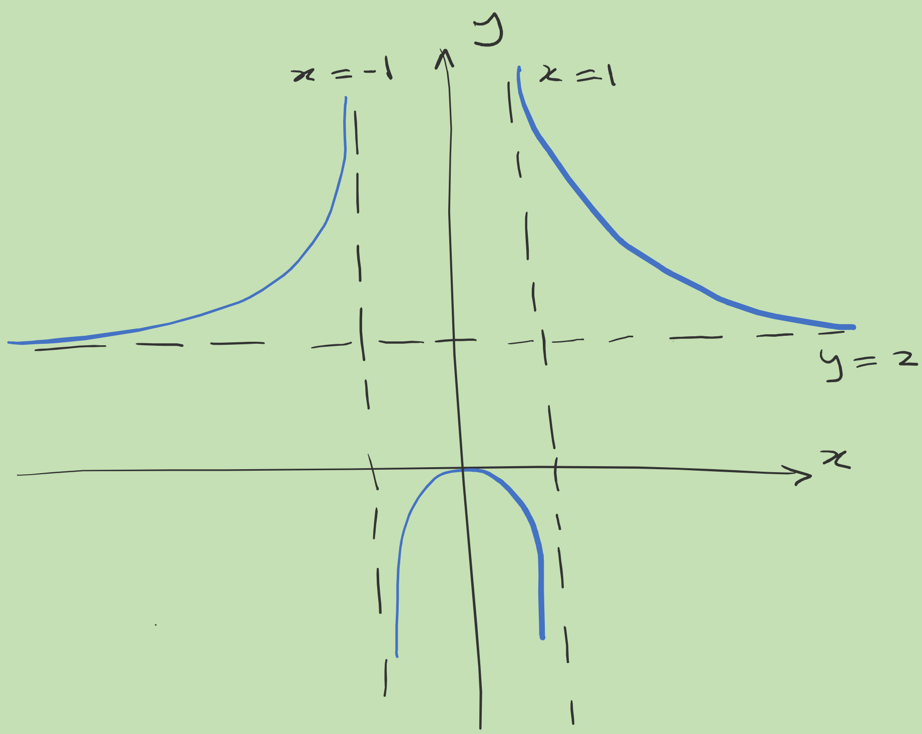
$$= \frac{4\cancel{(x^2-1)}\left[-(x^2-1)+4x^2\right]}{(x^2-1)^{\cancel{4}\,3}} = \frac{4(3x^2+1)}{(x^2-1)^3}$$

$f''(x) = 0 \Rightarrow 4(3x^2+1) = 0$ No real solution

f'' DNE $\Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$

	$(-\infty, -1)$ test point -2	$(-1, 1)$ test point 0	$(1, \infty)$ test point 2
$4(3x^2+1)$	$+$	$+$	$+$
$(x^2-1)^3$	$+$	$-$	$+$
$f''(x)$	$+$	$-$	$+$
$f(x)$	$\cup$	$\cap$	$\cup$

$x = \pm 1$ are inflection points



Example 2

Sketch the graph of $y = \ln(4 - x^2)$.

Solution

1. $4 - x^2 > 0$

$-x^2 > -4$

$x^2 < 4$

$-2 < x < 2$

$D = (-2, 2)$

تذكر

$x^2 > a \Rightarrow x > \sqrt{a}$ or $x < -\sqrt{a}$

$x^2 < a \Rightarrow -\sqrt{a} < x < \sqrt{a}$

2. y-intercept

$f(0) = \ln 4$

$(0, \ln 4)$

x-intercepts

$\ln(4 - x^2) = 0$

$4 - x^2 = 1 \Rightarrow x^2 = 4 - 1$

$x = \pm\sqrt{3}$

$(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$

3. $f(-x) = \ln(4 - (-x)^2)$

$= \ln(4 - x^2) = f(x)$

the function is symmetric about y-axis

4. $\lim\limits_{x \to \pm\infty} \ln(4 - x^2)$ DNE $\Rightarrow$ No HA

$\lim\limits_{x \to 2^-} \ln(4 - x^2)$

$= \ln(0^+) = -\infty$

$\lim\limits_{x \to -2^+} \ln(4 - x^2)$

$= \ln(0^+) = -\infty$

$x = 2$ and $x = -2$ are VA

5. $f'(x) = \dfrac{-2x}{4 - x^2}$

$-2x = 0 \Rightarrow x = 0$ the only cp

لاحظ $\boxed{f'(x)\ DNE}$ عند -2 و 2 لكنهم خارج domain

وبالتالي ليسوا critical points

6.

	-2	-1	0	1	2
$-2x$		$+$		$-$	
$4-x^2$		$+$		$+$	
$f'(x)$		$+$		$-$	
$f(x)$		$\nearrow$		$\searrow$	

$f(0) = \ln 4$ is a local maximum

7. $f''(x) = \dfrac{(4-x^2)\cdot -2 - (-2x)(-2x)}{(4-x^2)^2}$

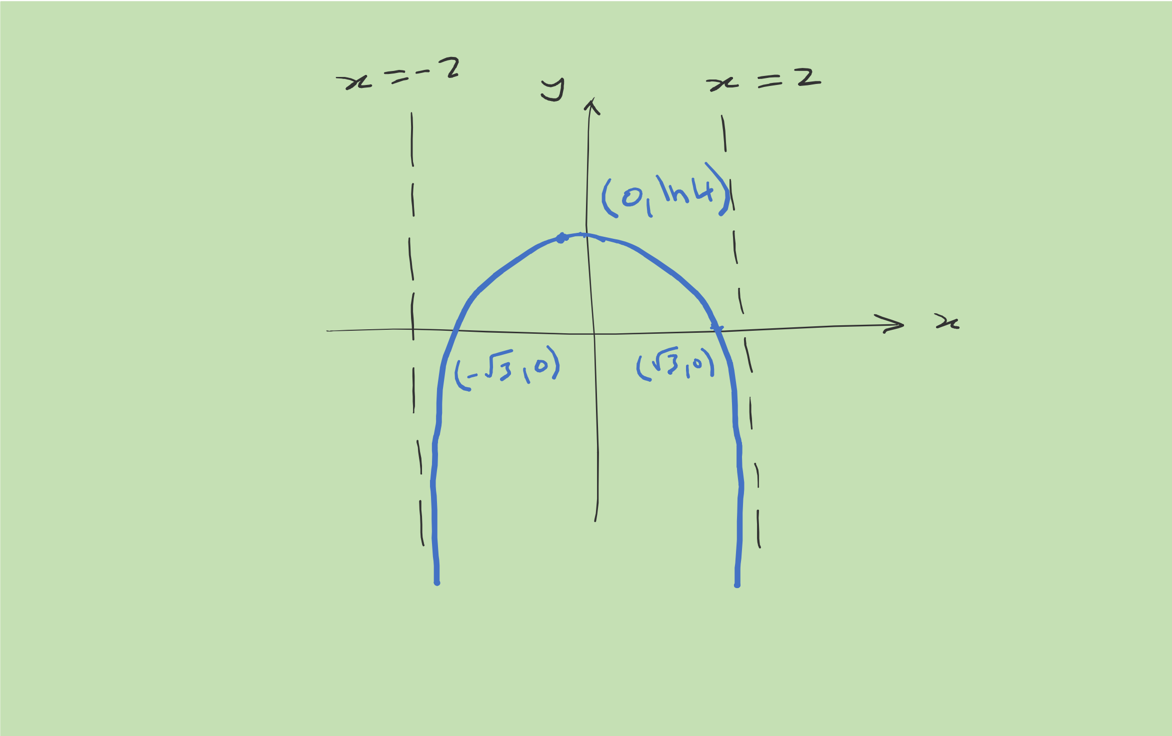
$$= \frac{-8 + 2x^2 - 4x^2}{(4-x^2)^2} = \frac{-8 - 2x^2}{(4-x^2)^2}$$

$-8 - 2x^2 = 0 \Rightarrow 2x^2 = -8$ No real solution

$f''(x)$ DNE for $x = \pm 2$

	-2	0	2
$-8-2x^2$		$-$	
$(4-x^2)^2$		$+$	
$f''(x)$		$-$	

$f(x)$ is Concave down


x = -2
y
x = 2
(0, ln4)
x
(-√3, 0)
(√3, 0)

Problems

- Sketch the graph of

(a) $f(x) = \frac{-2x}{x+1}$.

① Domain $\mathbb{R} \setminus \{-1\}$

② x-intercepts: $y = 0$ y-intercepts: $x = 0$

③ Symmetry:

$f(-x) =$

④ Asymptotes

$$\lim_{x \to \pm\infty} \frac{-2x}{x+1} \qquad \text{HA}$$

$$\lim_{x \to -1^+} \frac{-2x}{x+1}, \quad \lim_{x \to -1^-} \frac{-2x}{x+1} \qquad \text{VA}$$

⑤ Critical points $f'(x) \begin{cases} = 0 \\ \text{DNE} \end{cases}$

⑥ Intervals of Increase / Decrease $f'(x) \begin{cases} + \\ - \end{cases}$

Local <u>Extrema</u>

c.p ↗ ↘ max c.p ↘ ↗ min

⑦ Concavity and points of inflection $f'' \begin{cases} + & \cup \\ - & \cap \end{cases}$

(b) $f(x) = \frac{x^3}{x^2+1}$.

(c) $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$

1. $D = \mathbb{R}$

2. x-intercepts

$y \neq 0$

No x-intercepts

y-intercept

$f(0) = \frac{1}{2}(e^0 + e^0) = 1$

$(0, 1)$

3. $f(-x) = \frac{1}{2}(e^{-x} + e^{x}) = f(x)$

symmetric with respect to y-axis

4. No VA

$\lim\limits_{x \to \pm\infty} f(x)$ DNE $\Rightarrow$ No HA

5. $y' = \sinh x = \frac{1}{2}(e^x - e^{-x})$

$e^x - e^{-x} = 0 \Rightarrow e^x = e^{-x} \Rightarrow \frac{e^x}{e^{-x}} = 1$

$e^{2x} = 1 \Rightarrow x = 0$

and y' is defined everywhere

6.

	$-\infty$ -1 0	0 1 ∞
$y' = \frac{1}{2}(e^x - e^{-x})$	$-$	$+$
y	$\searrow$	$\nearrow$

$x = 0$ is a local min

$f(0) = 1$

7. $y'' = \cosh x = \frac{1}{2}(e^{x} + e^{-x})$

$\cosh x \neq 0$

and is defined everywhere $(\forall x \in \mathbb{R})$

therefore no cp for y''

$y'' = \frac{1}{2}(e^{x} + e^{-x})$

y