

Section 4.5 – Summary of Curve Sketching

- Guidelines for Sketching a Curve

خطوات رسم دالة

1. Find the domain

- جذر زوجي : ما عند الجذر موجب ≤ 0
- كسور : المقام $\neq 0$
- \ln, \log : ما بداخلها > 0

2. Intercepts

التقاطعات

i. x-intercepts

$$\Rightarrow y = 0 \Rightarrow (x, 0)$$

ii. y-intercepts

$$\Rightarrow x = 0 \Rightarrow (0, f(0))$$

3. Symmetry

التماثل

i. y-axis



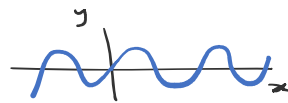
$$f(-x) = f(x) \text{ Even function}$$

ii. Origin



$$f(-x) = -f(x) \text{ Odd function}$$

iii. Periodicity



$$f(x+k) = f(x)$$

4. Asymptotes

i. Horizontal

$$\lim_{x \rightarrow \pm\infty} f(x) = a \Rightarrow y = a$$

ii. Vertical

$$\lim_{x \rightarrow b} f(x) = \pm\infty \Rightarrow x = b$$

5. Critical points

$$f'(x) = 0$$

6. Intervals of Increase or Decrease, Local Maximum and Minimum Values

$$f'(x) < \begin{matrix} + \\ - \end{matrix} \begin{matrix} \text{increase} \\ \text{decrease} \end{matrix}$$



7. Concavity and Points of Inflection

$$f'' \begin{cases} + \cup \\ - \cap \end{cases}$$

8. Sketch

Example 1

Use the guidelines to sketch the curve $y = \frac{2x^2}{x^2-1}$.

Solution

$$1. \mathcal{D} = \mathbb{R} \setminus \{-1, 1\}$$

$$2. 2x^2 = 0 \Rightarrow x = 0$$

$(0, 0)$ is the only intercept

$$3. f(-x) = \frac{2(-x)^2}{(-x)^2-1} = \frac{2x^2}{x^2-1} = f(x)$$

$\therefore f(x)$ is even \Rightarrow symmetric about y-axis

$$4. \lim_{x \rightarrow \infty} \frac{2x^2}{x^2-1} \cdot \frac{\div x^2}{\div x^2} = \lim_{x \rightarrow \infty} \frac{2}{1 - \frac{1}{x^2}} = 2$$

$$\lim_{x \rightarrow -\infty} \frac{2x^2}{x^2-1} \cdot \frac{\div x^2}{\div x^2} = \lim_{x \rightarrow -\infty} \frac{2}{1 - \frac{1}{x^2}} = 2$$

$y = 2$ is HA

$$x^2 - 1 = 0 \Rightarrow x = \pm 1$$

$$\lim_{x \rightarrow 1^-} \frac{2x^2}{x^2-1} = \frac{2}{0^-} = -\infty$$

$$\lim_{x \rightarrow -1^-} \frac{2x^2}{x^2-1} = \frac{2}{0^+} = \infty$$

$$\lim_{x \rightarrow 1^+} \frac{2x^2}{x^2-1} = \frac{2}{0^+} = \infty$$

$$\lim_{x \rightarrow -1^+} \frac{2x^2}{x^2-1} = \frac{2}{0^-} = -\infty$$

$x = 1$ and $x = -1$ are VA

$$5. f'(x) = \frac{(x^2-1) \cdot 4x - 2x^2(2x)}{(x^2-1)^2}$$

$$= \frac{\cancel{4x^3} - 4x - \cancel{4x^3}}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

$$f'(x) = 0 \Rightarrow -4x = 0 \Rightarrow x = 0$$

$$f'(x) \text{ DNE} \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$$

0, -1, 1 are critical points

6.

	-2	-1	-1/2	0	1/2	1	2	∞
$-4x$	+	⊖	+	0	-	⊖	-	
$(x^2-1)^2$	+		+		+		+	
$f'(x)$	+		+		-		-	
$f(x)$		↗	↗		↘		↘	

$f(0) = 0$ is a local max.

$$7. f''(x) = \frac{(x^2-1)^2 \cdot -4 - (-4x) \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4}$$

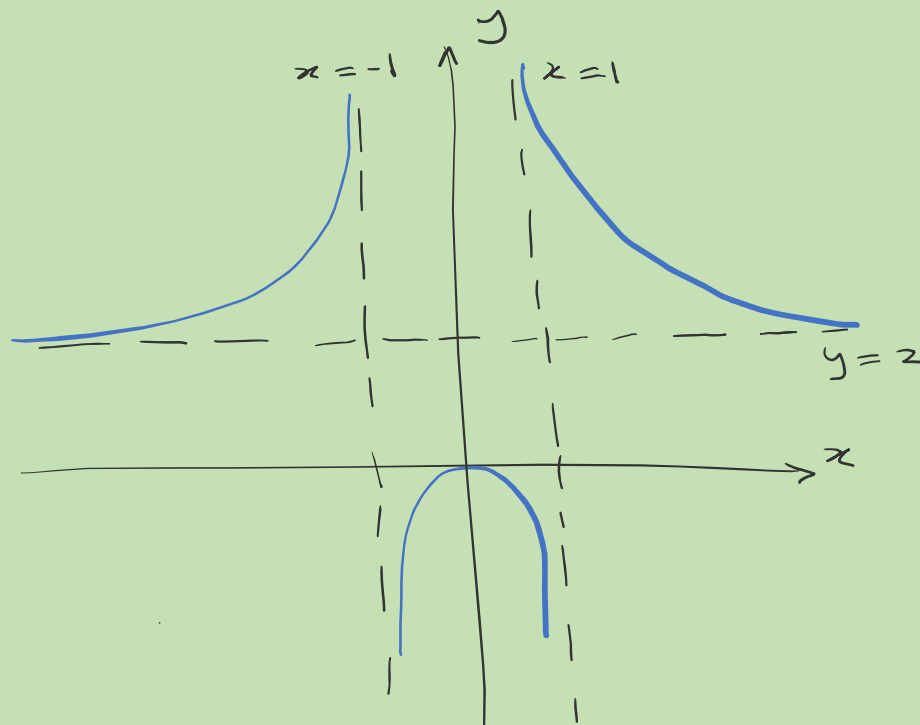
$$= \frac{4 \cancel{(x^2-1)} [-(x^2-1) + 4x^2]}{(x^2-1)^{\cancel{4}3}} = \frac{4(3x^2+1)}{(x^2-1)^3}$$

$$f''(x) = 0 \Rightarrow 4(3x^2+1) = 0 \quad \text{No real solution}$$

$$f'' \text{ DNE} \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$$

	$-\infty$	-2	-1	0	1	2	∞
$4(3x^2+1)$		+		+		+	
$(x^2-1)^3$		+		-		+	
$f''(x)$		+		-		+	
$f(x)$		U		∩		U	

$x = \pm 1$ are inflection points



Example 2

Sketch the graph of $y = \ln(4 - x^2)$.

Solution

$$1. \quad 4 - x^2 > 0$$

$$-x^2 > -4$$

$$x^2 < 4$$

$$-2 < x < 2$$

$$D = (-2, 2)$$

$$\begin{aligned} x^2 > a &\Rightarrow x > \sqrt{a} \text{ or } x < -\sqrt{a} \\ x^2 < a &\Rightarrow -\sqrt{a} < x < \sqrt{a} \end{aligned}$$

2. y -intercept

$$f(0) = \ln 4$$

$$(0, \ln 4)$$

x -intercepts

$$\ln(4 - x^2) = 0$$

$$4 - x^2 = 1 \Rightarrow x^2 = 4 - 1$$

$$x = \pm\sqrt{3}$$

$$(\sqrt{3}, 0) \text{ and } (-\sqrt{3}, 0)$$

$$3. \quad f(-x) = \ln(4 - (-x)^2)$$

$$= \ln(4 - x^2) = f(x)$$

the function is symmetric about y -axis

$$4. \quad \lim_{x \rightarrow \pm\infty} \ln(4 - x^2) \text{ DNE} \Rightarrow \text{NO HA}$$

$$\lim_{x \rightarrow 2^-} \ln(4 - x^2)$$

$$= \ln(0^+) = -\infty$$

$$\lim_{x \rightarrow -2^+} \ln(4 - x^2)$$

$$= \ln(0^+) = -\infty$$

$x = 2$ and $x = -2$ are VA

$$5. f'(x) = \frac{-2x}{4-x^2}$$

$$-2x = 0 \Rightarrow x = 0 \text{ the only cp}$$

domain $x \in]-2, 2[$ لكن 2 و -2 is $f'(x) \text{ DNE}$ نلاحظ
critical points ليسوا

	-2	-1	0	1	2
$-2x$		+		-	
$4-x^2$		+		+	
$f'(x)$		+		-	
$f(x)$		↗		↘	

$f(0) = \ln 4$ is a local maximum

$$7. f''(x) = \frac{(4-x^2) \cdot -2 - (-2x)(-2x)}{(4-x^2)^2}$$

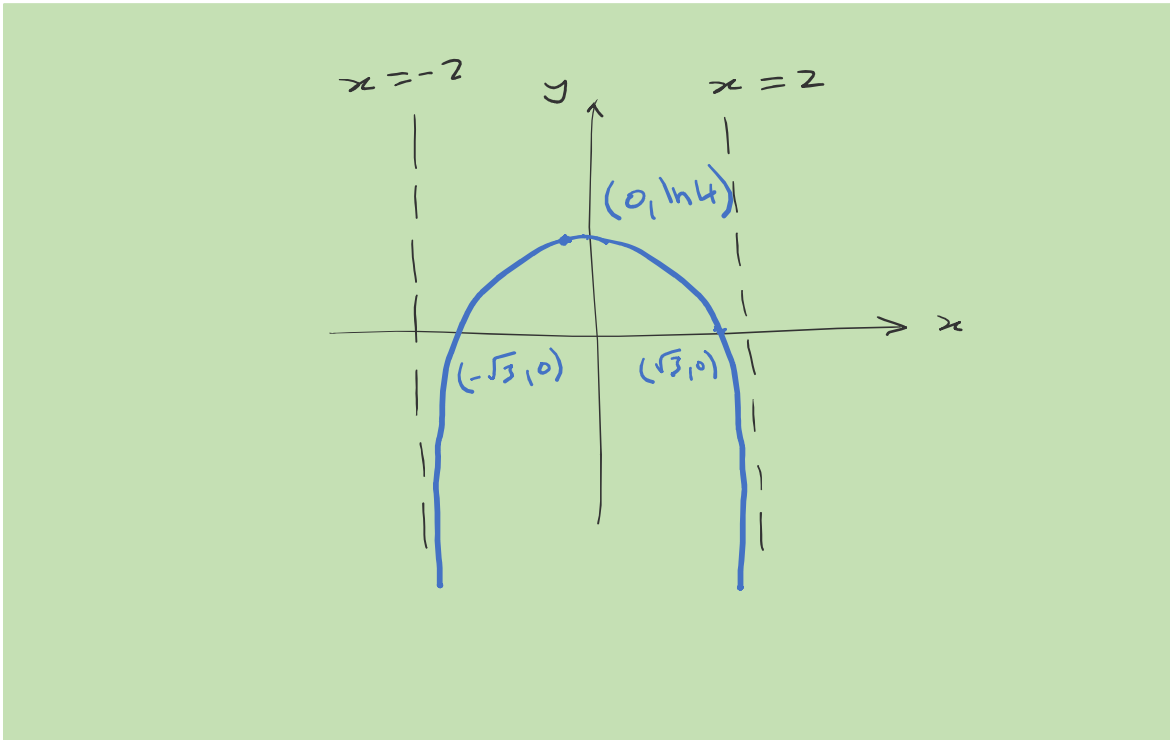
$$= \frac{-8 + 2x^2 - 4x^2}{(4-x^2)^2} = \frac{-8 - 2x^2}{(4-x^2)^2}$$

$$-8 - 2x^2 = 0 \Rightarrow 2x^2 = -8 \text{ No real solution}$$

$f''(x) \text{ DNE for } x = \pm 2$

	-2	0	2
$-8 - 2x^2$		-	
$(4-x^2)^2$		+	
$f''(x)$		-	

$f(x)$ is Concave
down



Problems

- Sketch the graph of

(a) $f(x) = \frac{-2x}{x+1}$.



(b) $f(x) = \frac{x^3}{x^2+1}$.

$$(c) y = \cosh x = \frac{1}{2}(e^x + e^{-x})$$

