Section 4.7 – Optimization Problems

- Steps In Solving Optimization Problems opt. خطوات حل مسائل

1. Identify

تعرف على
المعطيات
المطاليب
الشروط

2. Draw a Diagram

الرسم إن أمكن

3. Introduce Notation

اكتب رمز لكل شيء وعرّف الكمية المطلوب حلها، مثلا Q

4. Express Q in terms of some of the other symbols from Step 3.

اكتب Q بدلالة معطى واحد فقط

5. Find the absolute maximum or minimum

أوجد الحل

Closed Interval Method
لو فترة مغلقة

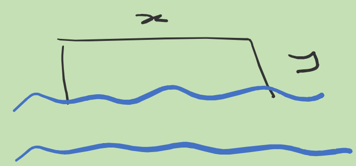
First Derivative Test
لو فترة مفتوحة

Example 1

A farmer has 1200 m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Solution

$x \geq 0 \quad , \quad y \geq 0 , \quad A \geq 0$



$A = xy \longrightarrow (1)$

$1200 = 2y + x$

$x = 1200 - 2y \longrightarrow (2)$

$A(y) = y(1200 - 2y) = 1200y - 2y^2$

$(1200 - 2y) \geq 0$

$y \geq 0 \qquad 1200 - 2y \geq 0$

$-2y \geq -1200$

$y \leq 600$

$D = [0, 600]$

$A'(y) = 1200 - 4$

$1200 - 4y = 0 \Rightarrow y = \frac{1200}{4} = 300$

$A(0) = 0$

$A(600) = 600(1200 - 2 \cdot 600) = 0$

$A(300) = 300(1200 - 2 \cdot 300) = 180{,}000 \text{ m}^2$

therefore the max. value is at $y = 300$

and $x = 1200 - 2(300) = 600 \text{ m}$

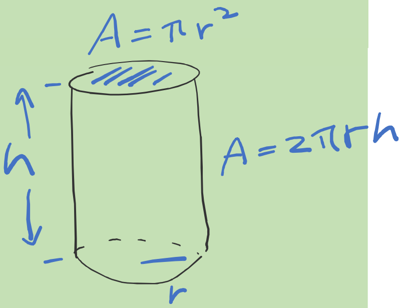
by Closed Interval Method

Example 2

Cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Solution

$r \geq 0 \ , \ h \geq 0 \ , \ A \geq 0$



$A = 2\pi r h + 2\pi r^2 \rightarrow (1)$

$V = 1L = 1000 \text{ cm}^3$

$1000 = \pi r^2 h \Rightarrow h = \dfrac{1000}{\pi r^2} \rightarrow (2)$

$A(r) = 2\cancel{\pi}\cancel{r}\left(\dfrac{1000}{\cancel{\pi}\cancel{r^2}}\right) + 2\pi r^2$

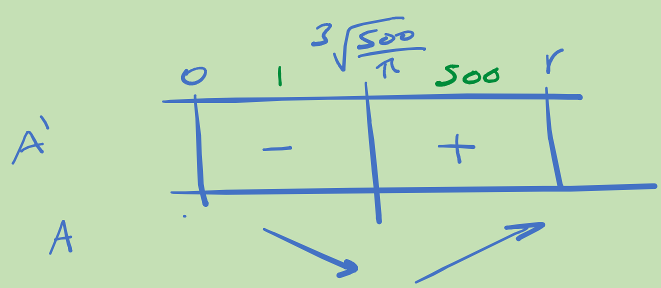
$= \dfrac{2000}{r} + 2\pi r^2 \qquad D = (0, r)$ ليست مغلقة

$A'(r) = -\dfrac{2000}{r^2} + 4\pi r$

$-\dfrac{2000}{r^2} + 4\pi r = 0 \Rightarrow 4\pi r^3 - 2000 = 0$

$r = \sqrt[3]{\dfrac{2000}{4\pi}} = \sqrt[3]{\dfrac{500}{\pi}}$

critical point



$A(r)$ has a local min at $r = \sqrt[3]{\dfrac{500}{\pi}}$

Since it is the only cp, therefore it is an absolute min.

and $h = \dfrac{1000}{\pi(500/\pi)^{2/3}}$

لاحظ

لا يمكن استخدام CIM في المثال السابق لأن فترة الحل
ليست مغلقة ، والبديل نستخدم :

First Derivative Test for Absolute Extreme Values

كالتالي :

① نتأكد أن f continuous ، ② لها cp واحدة فقط

If cp is local max $\Rightarrow$ absolute max.

local min $\Rightarrow$ absolute min.

Example 3

Find two numbers whose difference is 100 and whose product is a minimum.

Solution

$$a - b = 100 \Rightarrow a = 100 + b$$

$$f(a, b) = ab$$

$$f(b) = b(100 + b) \qquad b = (-\infty, \infty)$$

$$f'(b) = 100 + 2b$$

$$100 + 2b = 0 \Rightarrow b = \frac{-100}{2} = -50$$

	$-\infty$	-100	-50	100	∞
$f'(b)$		$-$		$+$	
$f(b)$		↘		↗	

$b = -50$ is a local min.
since it is the only cp
then it is an abs. min.
$a = 100 - 50 = 50$

Problems

- The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

$a > 0 \;,\; b > 0$

$a + b = 16 \Rightarrow b = 16 - a \longrightarrow ①$

$f(a,b) = a^2 + b^2 \longrightarrow ②$

$f(a) = a^2 + (16-a)^2$

$= a^2 + 256 - 32a + a^2$

$= 2a^2 - 32a + 256 \qquad a = (0, \infty)$

$f'(a) = 4a - 32$

$4a - 32 = 0 \Rightarrow a = \frac{32}{4} = 8$

	0	1	8	10	∞
$f'(a)$		$-$		$+$	
$f(a)$		↘		↗	

$a = 8$ is a local min.

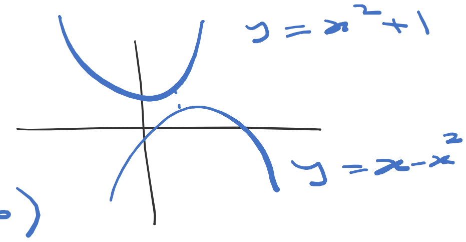
and since it is the only cp

therefore it is abs. min

$b = 16 - 8 = 8$

$f(8,8) = 8^2 + 8^2 = 128$

- What is the minimum vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$?



$$v(x) = x^2+1-(x-x^2)$$

$$= 2x^2 - x + 1 \qquad D = (0, \infty)$$

$$v'(x) = 4x - 1$$

$$4x - 1 = 0 \Rightarrow x = \frac{1}{4}$$

	0 $1/8$	$1/4$	1 ∞
$v'(x)$	$-$		$+$
$v(x)$	$\searrow$		$\nearrow$

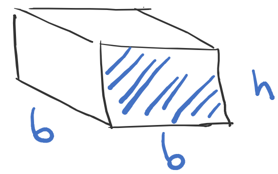
$x = 1/4$ is a local min.
since it is the only c p
there fore it is albs. min

$$v(x) = 2\left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right) + 1 = \frac{7}{8}$$

is the min. vertical distance

- If 1200 cm² of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

$b \geq 0 \quad , \quad h \geq 0$



$A = b^2 + 4bh$

$1200 = b^2 + 4bh$

$4bh = 1200 - b^2 \Rightarrow h = \dfrac{1200 - b^2}{4b} \longrightarrow ①$

$V = b^2 h \longrightarrow ②$

$V = b^{\cancel{2}}\left(\dfrac{1200 - b^2}{4\cancel{b}}\right) = \dfrac{1200b - b^3}{4}$

$V' = 300 - \frac{3}{4}b^2$ $\qquad D = (0, b)$

$300 - \frac{3}{4}b^2 = 0 \Rightarrow b^2 = 300 \cdot \frac{4}{3} = 400$

$b = 20$

	0	10	20	30	b
V'		+		−	
V		↗		↘	

$b = 20$ is a local max.

since it is the only cp

therefore it is the abs. max.

$V = \dfrac{1200 \cdot 20 - (20)^3}{4} = 4000 \text{ cm}^3$

- Find the point on the line $y = 2x + 3$ that is closest to the origin.

$d \geq 0$

$d = \sqrt{x^2 + y^2}$

$d(x) = \sqrt{x^2 + (2x+3)^2}$ $\qquad D = (-\infty, \infty)$

$= (x^2 + 4x^2 + 12x + 9)^{1/2}$

$= (5x^2 + 12x + 9)^{1/2}$

$d' = \frac{1}{2}(5x^2 + 12x + 9)^{-1/2}(10x + 12)$

$= \dfrac{10x + 12}{2(5x^2 + 12x + 9)}$

$d' = 0 \Rightarrow 10x + 12 = 0 \Rightarrow x = \frac{-12}{10} = \frac{-6}{5}$

	$-\infty$ -2	$-6/5$ 0 ∞
d'	$-$	$+$
$d(x)$	↘	↗

$x = \frac{-6}{5}$ is a local min

since it is the only cp

therefore it is the abs. min.

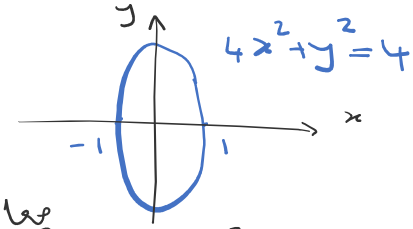
$y = 2\left(-\frac{6}{5}\right) + 3 = \frac{-12}{5} + \frac{15}{5} = \frac{3}{5}$

the point is $\left(-\frac{6}{5}, \frac{3}{5}\right)$

- Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

$$4x^2 + y^2 = 4$$

$$y^2 = 4 - 4x^2$$



معادلة ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$d = \sqrt{(x-1)^2 + y^2}$$

$$d(x) = \sqrt{(x-1)^2 + 4 - 4x^2} \qquad -1 \le x \le 1$$

$$D = [-1, 1]$$

$$= \sqrt{x^2 - 2x + 1 + 4 - 4x^2}$$

$$= \sqrt{-3x^2 - 2x + 5}$$

$$d' = \tfrac{1}{2}(-3x^2 - 2x + 5)^{-1/2}(-6x - 2)$$

$$= \frac{-6x - 2}{2(-3x^2 - 2x + 5)^{1/2}}$$

$$d' = 0 \Rightarrow -6x - 2 = 0 \Rightarrow 6x = -2 \Rightarrow x = -\frac{1}{3}$$

	-1	$-\frac{1}{4}$	$-\frac{1}{3}$	0	1
d'		$+$		$-$	
$d(x)$		$\nearrow$		$\searrow$	

$x = -\frac{1}{3}$ is local max.

since it is the only cp $\Rightarrow$ abs. max

$$y = \pm\sqrt{4 - 4\left(\tfrac{-1}{3}\right)^2} = \pm\sqrt{\tfrac{32}{9}} \Rightarrow \text{the points } \left(-\tfrac{1}{3}, \pm\sqrt{\tfrac{32}{9}}\right) \text{ are the farthest}$$