Section 4.9 – Antiderivatives

- Antiderivative حفظ

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

F هي antiderivative للدالة f اذا كانت مشتقة F هي f

- If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Example 1

Find the most general antiderivative of each of the following functions.
(a) $f(x) = \sin x$ (b) $f(x) = 1/x$ (c) $f(x) = x^n, \quad n \neq -1$

Solution

(a) $F(x) = -\cos x + C$

because $\frac{d}{dx}(-\cos x) = \sin x$

(b) $F(x) = \ln|x| + C$

because $\frac{d}{dx}[\ln|x|] = \begin{cases} \frac{1}{x} & x > 0 \\ \frac{-1}{-x} = \frac{1}{x} & x < 0 \end{cases}$

(c) $F(x) = \frac{x^{n+1}}{n+1} + C$

because $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{\cancel{(n+1)}x^n}{\cancel{n+1}} = x^n$

- Table of Antiderivative Formulas

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sec^2 x$	$\tan x$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
b^x	$\dfrac{b^x}{\ln b}$	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$

Example 2

Find all functions g such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$$

Solution

$$g'(x) = 4\sin x + \frac{2x^5}{x} - \frac{\sqrt{x}}{x}$$

$$= 4\sin x + 2x^4 - x^{-1/2}$$

$$g(x) = 4(-\cos x) + \frac{2x^5}{5} - \frac{x^{1/2}}{1/2} + C$$

$$= -4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + C$$

Example 3

Find f if $f'(x) = e^x + 20(1 + x^2)^{-1}$ and $f(0) = -2$.

Solution

$$f'(x) = e^x + \frac{20}{1+x^2}$$

$$f(x) = e^x + 20\tan^{-1}x + C$$

$$f(0) = e^0 + \cancel{20\tan^{-1}0} + C = -2$$

$$C = -2 - 1 = -3$$

$$\therefore f(x) = e^x + 20\tan^{-1}x - 3$$

Example 4

Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

Solution

$$f'(x) = \frac{12x^3}{3} + \frac{6x^2}{2} - 4x + C$$

$$= 4x^3 + 3x^2 - 4x + C$$

$$f(x) = \frac{4x^4}{4} + \frac{3x^3}{3} - \frac{4x^2}{2} + Cx + D$$

$$= x^4 + x^3 - 2x^2 + Cx + D$$

$$f(0) = 0 + 0 - 0 + 0 + D = 4 \Rightarrow D = 4$$

$$f(1) = 1 + 1 - 2 + C + 4 = 1 \Rightarrow C + 4 = 1 \Rightarrow C = -3$$

$$\therefore f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$

Problems

- Find the most general antiderivative of the function. (Check your answer by differentiation.)

(a) $f(x) = x^2 - 3x + 2$

$$F(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + C$$

because $\left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x + C\right]'$

$$= \frac{3x^2}{3} - \frac{6x}{2} + 2$$

$$= x^2 - 3x + 2$$

(b) $f(x) = x(12x + 8)$

$$f(x) = 12x^2 + 8x$$

$$F(x) = \frac{12x^3}{3} + \frac{8x^2}{2} + C = 4x^3 + 4x^2 + C$$

because $[4x^3 + 4x^2 + C]' = 12x^2 + 8x$

(c) $f(x) = e^2$

$$F(x) = e^2 x + C$$

because $[e^2 x + C]' = e^2$

(d) $f(x) = 3\sqrt{x} - 2\sqrt[3]{x}$

$$f(x) = 3x^{1/2} - 2x^{1/3}$$

$$F(x) = \frac{\cancel{3}\, x^{3/2}}{\cancel{3}/2} - \frac{\cancel{2}\, x^{4/3}}{{}^{2}\cancel{4}/3} + C$$

$$= 2x^{3/2} - \frac{3x^{4/3}}{2} + C$$

because $\left[2x^{3/2} - \frac{3x^{4/3}}{2} + C\right]'$

$$= \cancel{2}\cdot\frac{3}{\cancel{2}}x^{1/2} - \frac{\cancel{3}}{2}\cdot\frac{4}{\cancel{3}}x^{1/3}$$

$$= 3x^{1/2} - 2x^{1/3}$$

(e) $g(t) = \frac{1+t+t^2}{\sqrt{t}}$

$$g(t) = \frac{1}{\sqrt{t}} + \frac{t}{\sqrt{t}} + \frac{t^2}{\sqrt{t}} = t^{-1/2} + t^{1/2} + t^{3/2}$$

$$G(t) = \frac{t^{1/2}}{1/2} + \frac{t^{3/2}}{3/2} + \frac{t^{5/2}}{5/2} + C$$

$$= 2t^{1/2} + \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + C$$

because $\left[2t^{1/2} + \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + C\right]'$

$$= 2\cdot\frac{1}{2}t^{-1/2} + \frac{2}{3}\cdot\frac{3}{2}t^{1/2} + \frac{2}{5}\cdot\frac{5}{2}t^{3/2}$$

$$= t^{-1/2} + t^{1/2} + t^{3/2}$$

- Find f.

(a) $f''(x) = 1/x^2$

$$f''(x) = x^{-2}$$

$$f'(x) = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$$

$$f(x) = -\ln|x| + Cx + D = \begin{cases} -\ln x + C_1 x + D & x > 0 \\ -\ln(-x) + C_2 x + D & x < 0 \end{cases}$$

(b) $f''(\theta) = \sin\theta + \cos\theta, \quad f(0) = 3, \quad f'(0) = 4$

$$f'(\theta) = -\cos\theta + \sin\theta + C$$

$$f'(0) = -\cos 0 + \cancel{\sin 0}^{\,0} + C = 4$$

$$C = 4 + 1 = 5$$

$$f'(\theta) = -\cos\theta + \sin\theta + 5$$

$$f(\theta) = -\sin\theta - \cos\theta + 5\theta + D$$

$$f(0) = \cancel{-\sin 0}^{\,0} - \cos 0 + D = 3$$

$$D = 3 + 1 = 4$$

$$f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$$