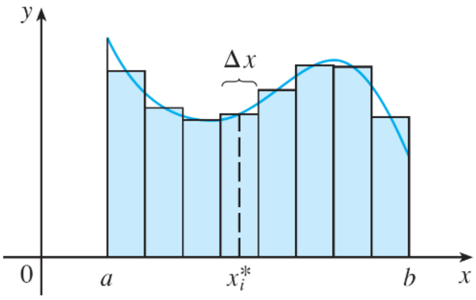
Section 5.2 – The Definite Integral

- Riemann sum



مساحة الشريحة = الارتفاع × العرض

$$R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x_i = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

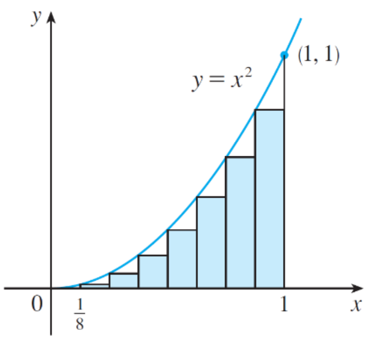
مجموع المساحات ← Sigma

ارتفاع الشريحة $f(x_i)$

$x_i = a + i\Delta x$

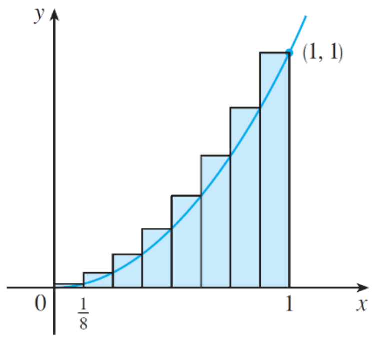
عرض الشريحة $\Delta x = \frac{b-a}{n}$

n ← عدد الشرائح



Left Sample points

$i = 0, 1, 2 \cdots, n$



Right sample Points

$i = 1, 2, 3, \cdots, n$

Example 1

Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

Solution

(a) $\Delta x = \dfrac{b-a}{n} = \dfrac{3}{6} = \dfrac{1}{2}$

$x_i = a + i\Delta x = 0 + i/2 = i/2$

i	x_i	$f(x_i)$
1	$1/2$	$(\frac{1}{2})^3 - 6(\frac{1}{2}) = 1/8 - 3 = \frac{-23}{8}$
2	$2/2 = 1$	$1^3 - 6 = -5$
3	$3/2$	$(\frac{3}{2})^3 - 6(\frac{3}{2}) = \frac{27}{8} - 9 = -\frac{45}{8}$
4	$4/2 = 2$	$2^3 - 6(2) = -4$
5	$5/2$	$(\frac{5}{2})^3 - 6(\frac{5}{2}) = \frac{125}{8} - 15 = \frac{5}{8}$
6	$6/2 = 3$	$3^3 - 6(3) = 9$

$$R_6 = \Delta x[f(x_1) + f(x_2) + \cdots + f(x_6)]$$

$$= \frac{1}{2}\left[\frac{-23}{8} - 5 - \frac{45}{8} - 4 + \frac{5}{8} + 9\right]$$

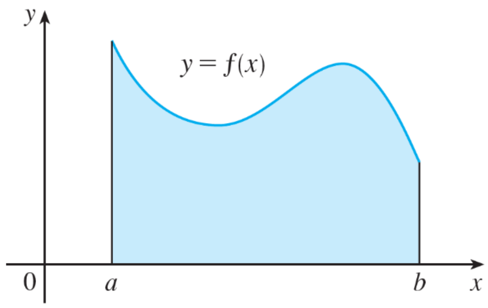
$$= \frac{1}{2} \cdot \frac{-63}{8} = \frac{-63}{16}$$

- If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

قانون التكامل

where $\qquad \Delta x = \frac{b-a}{n} \qquad$ and $\qquad x_i = a + i\Delta x$



تكامل الدالة $f(x)$ في الفترة من $x=a$ إلى $x=b$

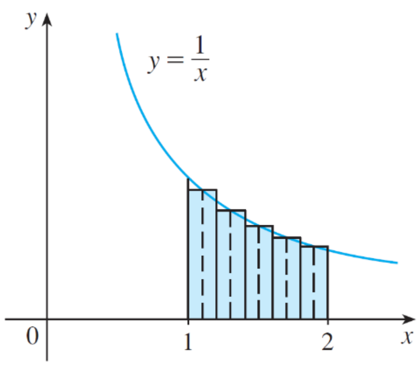
هو المساحة تحت منحنى الدالة

- The Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x[f(\bar{x}_1) + \cdots + f(\bar{x}_2)]$$

where $\qquad \Delta x = \frac{b-a}{n}$

and $\quad \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) =$ midpoint of $[x_{i-1}, x_i]$



Riemann sum يعطي قيمة تقريبية للتكامل

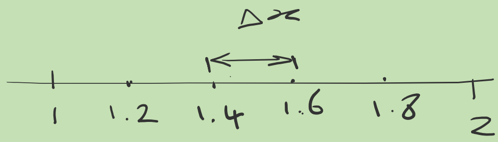
Example 2

Use the Midpoint Rule with $n = 5$ to approximate

$$\int_1^2 \frac{1}{x}\,dx$$

Solution

$$\Delta x = \frac{2-1}{5} = \frac{1}{5} = 0.2$$



i	$\bar{x}_i$
1	$\frac{1}{2}(1.0 + 1.2) = 1.1$
2	$\frac{1}{2}(1.2 + 1.4) = 1.3$
3	$\frac{1}{2}(1.4 + 1.6) = 1.5$
4	$\frac{1}{2}(1.6 + 1.8) = 1.7$
5	$\frac{1}{2}(1.8 + 2) = 1.9$

$+ \Delta x$

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x\left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)\right]$$

$$\approx 0.2\left[\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right]$$

$$\approx 0.6919$$

- Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant

2. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

3. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

Example 3

Use the properties of integrals to evaluate

$$\int_1^2 \left(4+\frac{1}{x}\right) dx$$

Solution

$$\int_1^2 \left(4+\frac{1}{x}\right) dx = \int_1^2 4\,dx + \int_1^2 \frac{1}{x}\,dx$$

$$= 4(2-1) + 0.6919$$

$$= 4.6919$$

Example 4

If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^{8} f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$

Solution

$$\int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$$

$$12 + \int_8^{10} f(x)\,dx = 17$$

$$\int_8^{10} f(x)\,dx = 17 - 12 = 5$$

- Comparison Properties of the Integral

6. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Example 5

Use Property 8 to estimate $\int_0^1 e^{-x^2}\,dx$.

Solution

$$f(x) = e^{-x^2}, \quad f(0) = 1, \quad f(1) = e^{-1} = \frac{1}{e}$$

$$\frac{1}{e}(1-0) \leq \int_0^1 e^{-x^2}\,dx \leq 1\,(1-0)$$

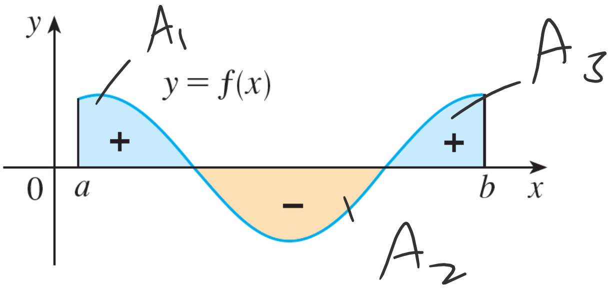
$$\frac{1}{e} \leq \int_0^1 e^{-x^2}\,dx \leq 1$$

لاحظ

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

- Area under the curve



$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$$

Example 6

Evaluate the following integrals by interpreting each in terms of areas.

(a) $\int_0^1 \sqrt{1-x^2}\,dx$

(b) $\int_0^3 (x-1)\,dx$

Solution

(a) $\int_0^1 \sqrt{1-x^2}\,dx = A_1$

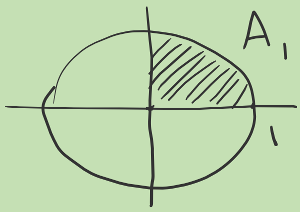
$= \frac{1}{4}\pi r^2$

$= \frac{\pi}{4}$

$y = \sqrt{1-x^2}$

$y^2 = 1-x^2$

$x^2+y^2=1$

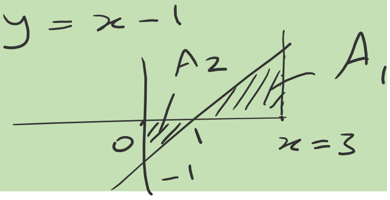


(b) $\int_0^3 (x-1) = A_1 - A_2$

$= \frac{1}{2}\cdot 2\cdot 2 - \frac{1}{2}\cdot 1\cdot 1$

$= 2 - \frac{1}{2} = \frac{3}{2}$

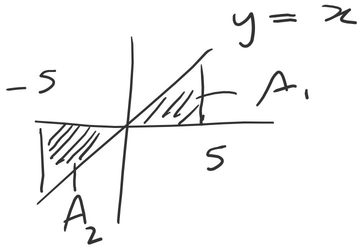
$y = x-1$



Problems

- Evaluate the integral by interpreting it in terms of areas.

(a) $\int_{-5}^{5}\left(x-\sqrt{25-x^2}\right)dx$



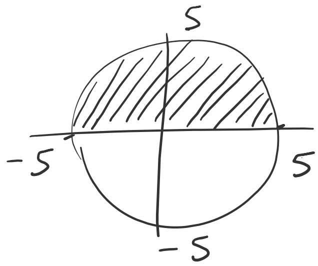
$$= \int_{-5}^{5} x - \int_{-5}^{5} \sqrt{25-x^2}\,dx$$

$$= \cancel{(A_1 - A_2)}^{0} - \frac{1}{2}\pi r^2$$

$$= -\frac{1}{2}\pi \cdot 25 = -\frac{25}{2}\pi$$

$y = \sqrt{25-x^2}$

$x^2 + y^2 = 25$



(b) $\int_{0}^{1}|2x-1|\,dx$

$y = |2x-1|$

$$= A_1 + A_2$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{2}\cdot 1$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

x-intercept

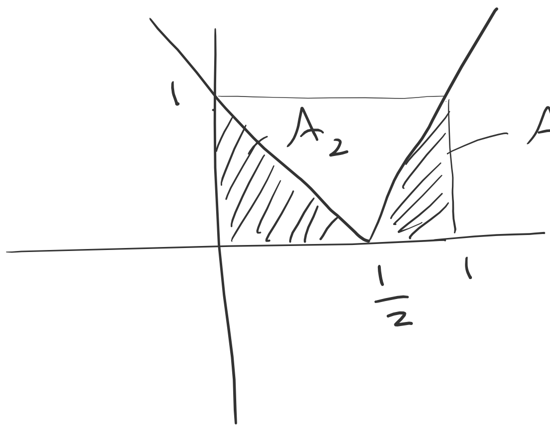
$0 = |2x-1|$

$x = 1/2$

y-intercept

$y = |-1|$

$= 1$



- Write as a single integral in the form $\int_a^b f(x)\,dx$:

$$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

$\because \int_{-2}^{-1} f(x)\,dx = -\int_{-1}^{-2} f(x)\,dx$

$\therefore \int_{-2}^{2} f(x) + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$

$= \int_{-1}^{-2} f(x)\,dx + \int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx$

$= \int_{-1}^{5} f(x)\,dx$

- If $\int_2^8 f(x)\,dx = 7.3$ and $\int_2^4 f(x)\,dx = 5.9$, find $\int_4^8 f(x)\,dx$.

$\int_{2}^{4} f(x)\,dx + \int_{4}^{8} f(x)\,dx = \int_{2}^{8} f(x)\,dx$

$5.9 + \int_{4}^{8} f(x)\,dx = 7.3$

$\int_{4}^{8} f(x)\,dx = 7.3 - 5.9 = 1.4$

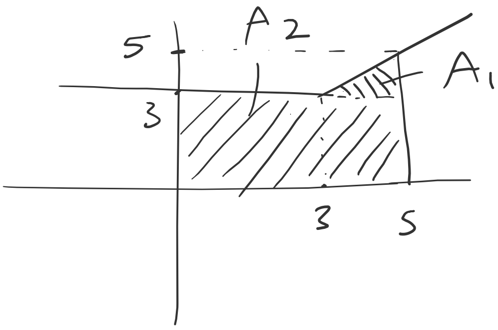
- Find $\int_0^5 f(x)\,dx$ if

$$f(x) = \begin{cases} 3 & \text{for} \quad x < 3 \\ x & \text{for} \quad x \geq 3 \end{cases}$$

$$\int_0^5 f(x)\,dx = A_1 + A_2$$

$$= \tfrac{1}{2} \cdot 2 \cdot 2 + 3 \cdot 5$$

$$= 2 + 15 = 17$$



- Use the properties of integrals to verify the inequality without evaluating the integrals.

(a) $\int_0^1 \sqrt{1+x^2}\,dx \le \int_0^1 \sqrt{1+x}\,dx$

$$x^2 \le x \qquad x \in [0,1]$$

$$1+x^2 \le 1+x$$

$$\sqrt{1+x^2} \le \sqrt{1+x}$$

$$\int_0^1 \sqrt{1+x^2}\,dx \le \int_0^1 \sqrt{1+x}\,dx$$

(b) $\frac{\pi}{12} \le \int_{\pi/6}^{\pi/3} \sin x\,dx \le \frac{\sqrt{3}\pi}{12}$

$$\frac{\pi}{6} \le x \le \frac{\pi}{3}$$

$$\sin\frac{\pi}{6} \le \sin x \le \sin\frac{\pi}{3}$$

$$\frac{1}{2} \le \sin x \le \frac{\sqrt{3}}{2}$$

$$\frac{1}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) \le \int_{\pi/6}^{\pi/3} \sin x\,dx \le \frac{\sqrt{3}}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$$

$$\frac{1}{2}\left(\frac{\pi}{6}\right) \le \int_{\pi/6}^{\pi/3} \sin x\,dx \le \frac{\sqrt{3}}{2}\left(\frac{\pi}{6}\right)$$

$$\frac{\pi}{12} \le \int_{\pi/6}^{\pi/3} \sin x\,dx \le \frac{\sqrt{3}}{12}\pi$$