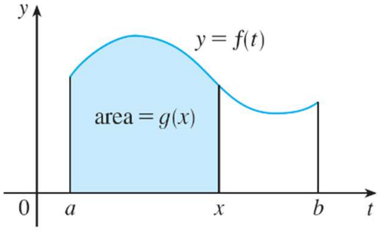
Section 5.3 – The Fundamental Theorem of Calculus

- The Fundamental Theorem of Calculus, Part 1

Suppose f is continuous on $[a, b]$.

If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.



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$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Example 1

Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$

Solution

$$f(t) = \sqrt{1+t^2}$$

$$g'(x) = f(x) = \sqrt{1+x^2}$$

- Chain rule for Fundamental Theorem of Calculus, part 1

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f\big(h(x)\big)\cdot h'(x) - f\big(g(x)\big)\cdot g'(x)$$

Example 2

Find $\frac{d}{dx}\int_1^{x^4} \sec t\,dt$

Solution

$$f(t) = \sec t \;,\; h(x) = x^4 ,\; g(x) = 1$$

$$\frac{d}{dx}\int_1^{x^4} \sec t\,dt = \sec(x^4)\cdot 4x^3 - \cancel{\sec(1)\cdot\overset{0}{(1)'}}$$

- **The Fundamental Theorem of Calculus, Part 2**

Suppose f is continuous on $[a, b]$.

$\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

antiderivative

لـ $F'(x)$ عن قيمة b

antiderivative

لـ $F'(x)$ عن قيمة a

Example 3

Evaluate the integral $\int_1^3 e^x\,dx$.

Solution

$$\int_1^3 e^x\,dx = e^x \Big|_1^3$$

$$= e^3 - e^1 = e^3 - e$$

Example 4

Evaluate $\int_3^6 \frac{dx}{x}$.

Solution

$$\int_3^6 \frac{1}{x}\,dx = \ln|x| \Big|_3^6$$

$$= \ln 6 - \ln 3$$

$$= \ln \frac{6}{3} = \ln 2$$

Example 5

Evaluate $\int_{-1}^{3}\frac{1}{x^2}\,dx$.

Solution

$\int_{-1}^{3}\frac{1}{x^2}\,dx$ DNE

because $\frac{1}{x^2}$ is not continuous

on $[-1,3]$

Example 6

Find the area under the parabola $y = x^2$ from 0 to 1.

Solution

$$A = \int_{0}^{1} x^2\,dx$$

$$= \frac{x^3}{3}\Big|_0^1$$

$$= \frac{1}{3} - 0 = \frac{1}{3}$$

Problems

- Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

(a) $h(u) = \int_0^u \frac{\sqrt{t}}{t+1}\, dt$

$$f(t) = \frac{\sqrt{t}}{t+1}$$

$$h'(u) = f(u)$$

$$= \frac{\sqrt{u}}{u+1}$$

(b) $R(y) = \int_y^0 t^3 \sin t\, dt$

$$R(y) = \int_y^0 t^3 \sin t\, dt = -\int_0^y t^3 \sin t\, dt$$

$$f(t) = t^3 \sin t$$

$$R'(y) = -y^3 \sin y$$

(c) $h(x) = \int_1^{e^x} \ln t \, dt$

$$f(t) = \ln t \quad , \quad j(x) = e^x \quad , \quad g(x) = 1$$

$$h'(x) = \ln e^x \cdot e^x - \overset{0}{\cancel{\ln 1 \cdot (1)'}}$$

$$= x e^x$$

(d) $y = \int_{1-3x}^{1} \frac{u^3}{1+u^2} \, du$

$$f(u) = \frac{u^3}{1+u^2} \quad , \quad h(x) = 1 \quad , \quad g(x) = 1 - 3x$$

$$y' = \overset{0}{\cancel{\frac{1^3}{1+1^2} \cdot (1)'}} - \frac{x^3}{1+x^2} \cdot -3$$

$$= \frac{3x^3}{1+x^2}$$

(e) $g(x) = \int_{1-2x}^{1+2x} t \sin t \, dt$

$f(t) = t\sin t, \quad j(x) = 1-2x, \quad h(x) = 1+2x$

$g'(x) = (1+2x)\sin(1+2x) \cdot 2 - (1-2x)\sin(1-2x) \cdot -2$

$= 2(1+2x)\sin(1+2x) + 2(1-2x)\sin(1-2x)$

(f) $F(x) = \int_{\sqrt{x}}^{2x} \arctan t \, dt$

$f(t) = \arctan t, \quad g(x) = \sqrt{x}, \quad h(x) = 2x$

$F'(x) = \arctan(2x) \cdot 2 - \arctan(\sqrt{x}) \cdot \frac{1}{2}x^{-\frac{1}{2}}$

$= 2\arctan(2x) - \frac{1}{2\sqrt{x}}\arctan(\sqrt{x})$

- Evaluate the integral.

(a) $\int_{\pi/6}^{\pi} \sin\theta \, d\theta$

$$= -\cos\theta \Big|_{\pi/6}^{\pi}$$

$$= -\cos\pi - (-\cos\pi/6)$$

$$= 1 + \frac{\sqrt{3}}{2} = \frac{2+\sqrt{3}}{2}$$

(b) $\int_{1}^{4} \frac{2+x^2}{\sqrt{x}} \, dx$

$$= \int_1^4 \left(\frac{2}{\sqrt{x}} + \frac{x^2}{\sqrt{x}}\right) dx$$

$$= \int_1^4 2x^{-1/2} \, dx + \int_1^4 x^{3/2} \, dx$$

$$= \frac{2x^{1/2}}{1/2} \Big|_1^4 + \frac{x^{5/2}}{5/2} \Big|_1^4$$

$$= 4\sqrt{x} \Big|_1^4 + \frac{2}{5}(\sqrt{x})^5 \Big|_1^4$$

$$= 4(2-1) + \frac{2}{5}(2^5 - 1)$$

$$= 4 + \frac{2}{5} \cdot 31 = \frac{20}{5} + \frac{62}{5} = \frac{82}{5}$$

(c) $\int_1^{18} \sqrt{\frac{3}{r}}\, dr$

$$= \int_1^{18} \sqrt{3}\, r^{-1/2}\, dr$$

$$= \sqrt{3}\, \frac{r^{1/2}}{1/2} \Big|_1^{18}$$

$$= 2\sqrt{3}\sqrt{r} \Big|_1^{18}$$

$$= 2\sqrt{3}\left(\sqrt{18} - 1\right)$$

(d) $\int_0^1 (x^e + e^x)\, dx$

$$= \frac{x^{e+1}}{e+1} \Big|_0^1 + e^x \Big|_0^1$$

$$= \left(\frac{1}{e+1} - 0\right) + \left(e^1 - e^0\right)$$

$$= \frac{1}{e+1} + e - 1$$

- Evaluate the integral and interpret it as a difference of areas. Illustrate with a sketch.

$$\int_{-1}^{2} x^3\,dx$$

$$\int_{-1}^{2} x^3\,dx = \frac{x^4}{4}\Big|_{-1}^{2}$$

$$= \frac{2^4}{4} - \left(\frac{(-1)^4}{4}\right)$$

$$= 4 - \frac{1}{4} = \frac{15}{4}$$

$$\int_{-1}^{2} x^3\,dx = \int_{-1}^{0} x^3\,dx + \int_{0}^{2} x^3\,dx$$

$$= A_1 - A_2$$