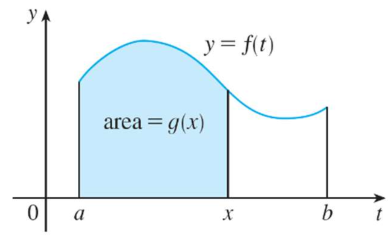
Section 5.3 – The Fundamental Theorem of Calculus

- The Fundamental Theorem of Calculus, Part 1

Suppose f is continuous on $[a, b]$.

If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.



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$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Example 1

Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$

Solution

$f(t) = \sqrt{1+t^2}$

$g'(x) = f(x) = \sqrt{1+x^2}$

- Chain rule for Fundamental Theorem of Calculus, part 1

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f\big(h(x)\big)\cdot h'(x) - f\big(g(x)\big)\cdot g'(x)$$

Example 2

Find $\frac{d}{dx}\int_1^{x^4} \sec t\,dt$

Solution

$f(t) = \sec t$, $h(x) = x^4$, $g(x) = 1$

$\frac{d}{dx}\int_1^{x^4} \sec t\,dt = \sec(x^4)\cdot 4x^3 - \sout{\sec(1)\cdot(1)'}$ (0)

- **The Fundamental Theorem of Calculus, Part 2**

Suppose f is continuous on $[a, b]$.

$\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

antiderivative
لـ $F'(x)$ مشتقة b

antiderivative
لـ $F'(x)$ مشتقة a

Example 3

Evaluate the integral $\int_1^3 e^x\,dx$.

Solution

$$\int_1^3 e^x\,dx = e^x \Big|_1^3$$

$$= e^3 - e^1 = e^3 - e$$

Example 4

Evaluate $\int_3^6 \frac{dx}{x}$.

Solution

$$\int_3^6 \frac{1}{x}\,dx = \ln|x| \Big|_3^6$$

$$= \ln 6 - \ln 3$$

$$= \ln \frac{6}{3} = \ln 2$$

Example 5

Evaluate $\int_{-1}^{3} \frac{1}{x^2} dx$.

Solution

$\int_{-1}^{3} \frac{1}{x^2} dx$ DNE

because $\frac{1}{x^2}$ is not continuous

on $[-1, 3]$

Example 6

Find the area under the parabola $y = x^2$ from 0 to 1.

Solution

$$A = \int_0^1 x^2 \, dx$$

$$= \left. \frac{x^3}{3} \right|_0^1$$

$$= \frac{1}{3} - 0 = \frac{1}{3}$$

Problems

- Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

(a) $h(u) = \int_0^u \frac{\sqrt{t}}{t+1} dt$

(b) $R(y) = \int_y^0 t^3 \sin t \, dt$

(c) $h(x) = \int_1^{e^x} \ln t \, dt$

(d) $y = \int_{1-3x}^{1} \frac{u^3}{1+u^2} \, du$

(e) $g(x) = \int_{1-2x}^{1+2x} t \sin t \, dt$

(f) $F(x) = \int_{\sqrt{x}}^{2x} \arctan t \, dt$

- **Evaluate the integral.**

(a) $\int_{\pi/6}^{\pi} \sin\theta \, d\theta$

(b) $\int_{1}^{4} \frac{2+x^2}{\sqrt{x}} dx$

(c) $\int_1^{18} \sqrt{\frac{3}{r}}\, dr$

(d) $\int_0^1 (x^e + e^x)\, dx$

- Evaluate the integral and interpret it as a difference of areas. Illustrate with a sketch.

$$\int_{-1}^{2} x^3 \, dx$$