Section 5.4 – Indefinite Integrals and the Net Change Theorem

- Indefinite integral ≡ Antiderivative notation

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

$\int x^2\,dx = \frac{x^3}{3} + C$ because $\frac{d}{dx}\left(\frac{x^3}{3} + C\right) = \frac{\cancel{3}x^2}{\cancel{3}} = x^2$

حفظ

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int k\,dx = kx + C$	
$\int x^n\,dx = \frac{x^{n+1}}{n+1} \qquad (n \neq 1)$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int b^x\,dx = \frac{b^x}{\ln b} + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
$\int \sinh x\,dx = \cosh x + C$	$\int \cosh x\,dx = \sinh x + C$

Example 1

Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\, dx$$

Solution

$$= \int 10x^4\, dx - \int 2\sec^2 x\, dx$$

$$= 10\int x^4\, dx - 2\int \sec^2 x\, dx$$

$$= 10\,\frac{x^5}{5} - 2\tan x + C$$

$$= 2x^5 - 2\tan x + C$$

Example 2

Evaluate

$$\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

Solution

$$= \int \frac{1}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta}\, d\theta$$

$$= \int \csc\theta \cdot \cot\theta\, d\theta$$

$$= -\csc\theta + C$$

Example 3

Evaluate

$$\int_0^3 (x^3 - 6x)\,dx$$

Solution

$$= \frac{x^4}{4} - \frac{6x^2}{2} \Big|_0^3$$

$$= \left[\frac{3^4}{4} - 3(3)^2\right] - \left[\frac{0}{4} - \frac{0}{2}\right]$$

$$= \frac{81}{4} - 27 = -\frac{27}{4}$$

Example 4

Find

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx$$

Solution

$$= \frac{2x^4}{4} - \frac{6x^2}{2} + 3\tan^{-1} x \Big|_0^2$$

$$= \frac{2^4}{2} - 3(2)^2 + 3\tan^{-1} 2 - 0$$

$$= 8 - 12 + 3\tan^{-1} 2 = -4 + 3\tan^{-1} 2$$

Example 5

Find

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$$

Solution

$$\int_1^9 \frac{2t^2}{t^2} + \frac{t^2\sqrt{t}}{t^2} - \frac{1}{t^2}\,dt$$

$$= \int_1^9 2 + \sqrt{t} - t^{-2}\,dt$$

$$= 2t + \frac{t^{3/2}}{3/2} - \frac{t^{-1}}{-1}\Bigg|_1^9$$

$$= \left[2\cdot 9 + \frac{2}{3}(9)^{3/2} + \frac{1}{9}\right] - \left[2 + \frac{2}{3} + 1\right]$$

$$= \left[18 + \frac{2}{\cancel{3}}\cdot \overset{9}{\cancel{27}} + \frac{1}{9}\right] - \left[\frac{9}{3} + \frac{2}{3}\right]$$

$$= \left[\frac{324}{9} + \frac{1}{9}\right] - \left[\frac{11}{3}\right]$$

$$= \frac{325}{9} - \frac{11}{3}$$

$$= \frac{325}{9} - \frac{33}{9} = \frac{292}{9}$$

Problems

- Verify by differentiation that the formula is correct.

$$\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

$$\frac{d}{dx}\left(\frac{1}{2}x + \frac{1}{4}\sin 2x + C\right)$$

$$= \frac{1}{2} + \frac{1}{4}\cdot 2\cos 2x$$

$$= \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$= \frac{1}{2}(1+\cos 2x) = \cos^2 x$$

- Find the general indefinite integral.

(a) $\int \sqrt[4]{x^5}\, dx$

$$\int x^{5/4}\, dx$$

$$= \frac{x^{9/4}}{9/4} + C$$

$$= \frac{4}{9} x^{9/4} + C$$

(b) $\int \left(u^6 - 2u^5 - u^3 + \frac{2}{7}\right) du$

$$= \frac{u^7}{7} - \frac{2u^6}{6} - \frac{u^4}{4} + \frac{2}{7}u + C$$

(c) $\int v(v^2+2)^2\, dv$

$$\int v\left(v^4 + 4v^2 + 4\right) dv$$

$$= \int v^5 + 4v^3 + 4v$$

$$= \frac{v^6}{6} + \frac{4v^4}{4} + \frac{4v^2}{2} + C$$

$$= \frac{v^6}{6} + v^4 + 2v^2 + C$$

(d) $\int \left(x^2+1+\frac{1}{x^2+1}\right)dx$

$$= \frac{x^3}{3} + x + \tan^{-1} x + C$$

(e) $\int \left(\frac{1+r}{r}\right)^2 dr$

$$= \int \frac{(1+r)^2}{r^2}\, dr$$

$$= \int \frac{1+2r+r^2}{r^2}\, dr$$

$$= \int \frac{1}{r^2} + \frac{2r}{r^2} + \frac{r^2}{r^2}\, dr$$

$$= \int r^{-2} + 2\cdot\frac{1}{r} + 1\, dr$$

$$= \frac{r^{-1}}{-1} + 2\ln|r| + r + C$$

$$= -\frac{1}{r} + 2\ln|r| + r + C$$

- Evaluate the integral.

(a) $\int_0^3 (1 + 6w^2 - 10w^4)\,dw$

$$= w + \frac{6w^3}{3} - \frac{10w^5}{5} \Big|_0^3$$

$$= 3 + 2(3)^3 - 2(3)^5 - 0$$

$$= -429$$

(b) $\int_1^2 \left(\frac{1}{x^2} - \frac{4}{x^3}\right) dx$

$$= \int_1^2 x^{-2} - 4x^{-3}\,dx$$

$$= \frac{x^{-1}}{-1} - \frac{4x^{-2}}{-2} \Big|_1^2$$

$$= -\frac{1}{x} + \frac{2}{x^2} \Big|_1^2$$

$$= \cancel{\left(-\frac{1}{2} + \frac{2}{4}\right)}^{0} - (-1 + 2) = -1$$

(c) $\int_0^1 (5x - 5^x)\,dx$

$$= \frac{5x^2}{2} - \frac{5^x}{\ln 5} \Big|_0^1$$

$$= \left(\frac{5}{2} - \frac{5}{\ln 5}\right) - \left(0 - \frac{1}{\ln 5}\right)$$

$$= \frac{5}{2} - \frac{5}{\ln 5} + \frac{1}{\ln 5} = \frac{5}{2} - \frac{4}{\ln 5}$$

(d) $\int_0^2 |2x-1|\,dx$

$2x-1 \geq 0$

$$|2x-1| = \begin{cases} 2x-1 & x \geq \frac{1}{2} \\ -(2x-1) & x < \frac{1}{2} \end{cases}$$

$2x-1 < 0$

therefore $\int_0^2 |2x-1|\,dx$

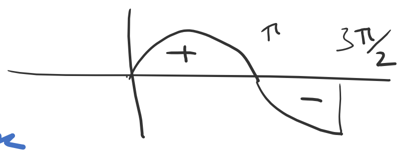
$$= \int_0^{1/2} 1-2x \quad dx + \int_{1/2}^{2} 2x-1 \; dx$$

$$= \left[x - \frac{\cancel{2}x^2}{\cancel{2}}\right]_0^{1/2} + \left[\frac{\cancel{2}x^2}{\cancel{2}} - x\right]_{1/2}^{2}$$

$$= \left(\tfrac{1}{2} - \tfrac{1}{4}\right) + \left[(4-2) - \left(\tfrac{1}{4} - \tfrac{1}{2}\right)\right]$$

$$= \frac{1}{4} + \left(2 - \left(-\frac{1}{4}\right)\right) = 2 + \frac{1}{2} + \frac{5}{2}$$

(e) $\int_0^{3\pi/2} |\sin x|\,dx$



$$= \int_0^{\pi} \sin x \; dx + \int_{\pi}^{3\pi/2} -\sin x \; dx$$

$$= -\cos x \Big|_0^{\pi} - (-\cos x)\Big|_{\pi}^{3\pi/2}$$

$$= -\cos\pi - (-\cos 0) + \left(\overset{0}{\cancel{\cos\tfrac{3\pi}{2}}} - \cos\pi\right)$$

$$= -(-1) - (-1) + (0 - (-1))$$

$$= 2 + 1 = 3$$