Section 5.5 – The Substitution Rule

- Integrating non-standard functions

$$\int 2x\sqrt{1+x^2}\,dx$$

$g(x)$ $f(x)$

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نحتاج إلى قاعدة جديدة

- **The substitution Rule**

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)dx = \int f(u)\,du$$

$\because$ $2x$ هي مشتقة $1+x^2$

$\therefore\ u = 1 + x^2 \Rightarrow du = 2x\,dx$

ونعيد تعريف التكامل بدلالة u

$$\int \sqrt{u}\,du = \frac{u^{3/2}}{3/2} + C = \frac{(1+x^2)^{3/2}}{3/2} + C$$

Example 1

Find

$$\int x^3 \cos(x^4 + 2)\,dx$$

Solution

$u = x^4 + 2 \Rightarrow du = 4x^3\,dx \Rightarrow \frac{1}{4}du = x^3$

$\frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C$

$= \frac{1}{4}\sin(x^4+2) + C$

Example 2

Evaluate

$$\int \sqrt{2x+1}\,dx$$

Solution

$$u = 2x+1 \Rightarrow du = 2\,dx \Rightarrow \frac{1}{2}du = dx$$

$$\frac{1}{2}\int \sqrt{u}\,du = \cancel{\frac{1}{2}}\,\frac{u^{3/2}}{3/\cancel{2}} + C$$

$$= \frac{(2x+1)^{3/2}}{3} + C$$

Example 3

Find

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx$$

Solution

$$u = 1-4x^2 \Rightarrow du = -8x\,dx \Rightarrow -\frac{1}{8}du = x\,dx$$

$$-\frac{1}{8}\int \frac{du}{\sqrt{u}} = -\frac{1}{8}\,\frac{u^{1/2}}{1/2} + C$$

$$= -\frac{2}{8}\sqrt{u} + C$$

$$= -\frac{1}{4}\sqrt{1-4x^2} + C$$

Example 4

Calculate

$$\int e^{5x}\,dx$$

Solution

$$u = 5x \Rightarrow du = 5\,dx \Rightarrow \frac{1}{5}du = dx$$

$$\frac{1}{5}\int e^{u}\,du = \frac{1}{5}e^{u} + C$$

$$= \frac{1}{5}e^{5x} + C$$

Example 5

Calculate

$$\int \sqrt{1+x^2}x^5\,dx$$

Solution

$$u = 1+x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$$

$$x^2 = u-1 \Rightarrow x^4 = (u-1)^2$$

$$\int \sqrt{1+x^2}\cdot x^4\cdot x\,dx = \frac{1}{2}\int \sqrt{u}\,(u-1)^2\,du$$

$$= \frac{1}{2}\int \sqrt{u}\,(u^2-2u+1)\,du = \frac{1}{2}\int u^{5/2} - 2u^{3/2} + u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{u^{7/2}}{7/2} - 2\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C$$

$$= \frac{(1+x^2)^{7/2}}{7} - \frac{2(1+x^2)^{5/2}}{5} + \frac{(1+x^2)^{3/2}}{3} + C$$

- **The Substitution Rule for Definite Integrals**

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

نحتاج نحسب حدود جديدة للتكامل

Example 6

Evaluate

$$\int_0^4 \sqrt{2x+1}\,dx$$

Solution

$u = 2x + 1 \Rightarrow du = 2\,dx \Rightarrow \frac{1}{2}du = dx$

$u(4) = 2(4) + 1 = 9$

$u(0) = 2(0) + 1 = 1$

$$\frac{1}{2}\int_1^9 \sqrt{u}\;du$$

$$= \frac{1}{\cancel{2}} \frac{u^{3/2}}{3/\cancel{2}}\Big|_1^9$$

$$= \frac{u^{3/2}}{3}\Big|_1^9$$

$$= \frac{9^{3/2}}{3} - \frac{1^{3/2}}{3} = \frac{27}{3} - \frac{1}{3}$$

$$= 9 - \frac{1}{3} = \frac{26}{3}$$

Example 7

Evaluate

$$\int_1^2 \frac{dx}{(3-5x)^2}$$

Solution

$$u = 3-5x \Rightarrow du = -5\,dx \Rightarrow \frac{1}{-5}du = dx$$

$$u(2) = 3-5(2) = -7$$

$$u(1) = 3-5(1) = -2$$

$$-\frac{1}{5}\int_{-2}^{-7} \frac{1}{u^2}\,du = \cancel{-}\frac{1}{5}\left[\frac{u^{-1}}{\cancel{-}1}\right]_{-2}^{-7}$$

$$= \frac{1}{5}\left[\frac{1}{-7} - \frac{1}{-2}\right] = \frac{1}{5}\left(\frac{1}{2} - \frac{1}{7}\right)$$

$$= \frac{1}{5}\left(\frac{7-2}{14}\right) = \frac{1}{14}$$

Example 8

Evaluate

$$\int_1^e \frac{\ln x}{x}\,dx$$

Solution

$$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$$

$$u(e) = \ln e = 1$$

$$u(1) = \ln 1 = 0$$

$$\int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$$

- Symmetry

Suppose f is continuous on $[-a, a]$.

$$\int_{-a}^{a} f(x)\,dx$$

f is odd	f is even
$= 0$	$= 2\int_0^a f(x)\,dx$

Example 9

Evaluate

$$\int_{-2}^{2} (x^6 + 1)\,dx$$

Solution

$f(x) = x^6 + 1$

$f(-x) = (-x)^6 + 1 = x^6 + 1 = f(x)$

$\therefore \int_{-2}^{2} (x^6+1)\,dx = 2\int_0^2 (x^6+1)\,dx$

$= 2\left[\frac{x^7}{7} + x\right]_0^2$

$= 2\left[\left(\frac{2^7}{7} + 2\right) - 0\right]$

$= \frac{2^8}{7} + 4$

$= \frac{256}{7} + \frac{28}{7} = \frac{284}{7}$

Problems

- Evaluate the integral by making the given substitution.

(a) $\int xe^{-x^2}\,dx, \quad u = -x^2$

$$du = -2x\,dx \Rightarrow -\frac{1}{2}du = x\,dx$$

$$-\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C$$

$$= -\frac{1}{2}e^{-x^2} + C$$

(b) $\int \sin^2\theta\cos\theta\,d\theta, \quad u = \sin\theta$

$$du = \cos\theta\,d\theta$$

$$\int u^2\,du = \frac{u^3}{3} + C$$

$$= \frac{\sin^3\theta}{3} + C$$

- Evaluate the indefinite integral.

(a) $\int e^{-5r}\,dr$

$u = -5r \Rightarrow du = -5\,dr \Rightarrow \frac{1}{-5}\,du = dr$

$-\frac{1}{5}\int e^{u}\,du = -\frac{1}{5}e^{u} + C$

$= -\frac{1}{5}e^{-5r} + C$

(b) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

$u = \sqrt{x} \Rightarrow du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx \quad \times 2$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

$2\int \sin u\,du = -2\sin u + C$

$= -2\sin\sqrt{x} + C$

(c) $\int e^{\cos t}\sin t\,dt$

$u = \cos t \Rightarrow du = -\sin t\,dt \Rightarrow -du = \sin t\,dt$

$-\int e^{u}\,du = -e^{u} + C$

$= -e^{\cos t} + C$

(d) $\int \frac{x}{1+x^4} dx$

$$u = x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2} du = x\,dx$$

$$\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1} u + C$$

$$= \frac{1}{2}\tan^{-1} x^2 + C$$

(e) $\int \frac{1+x}{1+x^2} dx$

$$= \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx$$

$$= \tan^{-1} x + \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \tan^{-1} x + \frac{1}{2}\ln|u| + C$$

$$= \tan^{-1} x + \frac{1}{2}\ln(1+x^2) + C$$

$$\boxed{\begin{aligned} u &= 1+x^2 \\ du &= 2x\,dx \\ \tfrac{1}{2}du &= x\,dx \end{aligned}}$$

$1+x^2 > 0$

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- Evaluate the definite integral.

(a) $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

$u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2}\,dx \Rightarrow -du = \frac{1}{x^2}\,dx$

$u(2) = \frac{1}{2}$

$u(1) = 1$

$-\int_1^{1/2} e^u\,du = -\,e^u \Big|_1^{1/2}$

$= -\left[e^{1/2} - e^1\right]$

$= e - \sqrt{e}$

(b) $\int_0^{\pi/2} \cos x \sin(\sin x)\,dx$

$u = \sin x \Rightarrow du = \cos x\,dx$

$u(0) = \sin 0 = 0$

$u(\pi/2) = \sin \pi/2 = 1$

$\int_0^1 \sin u\,du = -\cos u \Big|_0^1$

$= -\left[\cos 1 - \overset{1}{\cancel{\cos 0}}\right]$

$= 1 - \cos 1$

(c) $\int_0^1 \frac{e^y+1}{e^y+y} dy$

$u = e^y + y \Rightarrow du = (e^y + 1)\, dy$

$u(1) = e + 1$

$u(0) = 1$

$$\int_1^{e+1} \frac{1}{u}\, du = \ln|u| \Big|_1^{e+1}$$

$$= \ln(e+1) - \ln 1$$

$$= \ln\left(\frac{e+1}{1}\right) = \ln(e+1)$$

(d) $\int_{-1}^1 \frac{\tan x}{1+x^2+x^4} dx$

$$f(x) = \frac{\tan x}{1 + x^2 + x^4}$$

$$f(-x) = \frac{\tan(-x)}{1 + (-x)^2 + (-x)^4}$$

$$= \frac{-\tan x}{1 + x^2 + x^4} = -f(x)$$

therefore $\int_{-1}^{1} f(x)\, dx = 0$