

Section 5.5 – The Substitution Rule

- Integrating non-standard functions

$$\int 2x\sqrt{1+x^2} dx$$

$$g(x) \quad f(x)$$

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نحتاج إلى قاعدة جديدة

- The substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I , then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

$$\text{مثال: } 2x \text{ هي مشتقة } 1+x^2$$

$$\text{مثال: } u = 1+x^2 \Rightarrow du = 2x dx$$

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$$\int \sqrt{u} du = \frac{u^{3/2}}{3/2} + C = \frac{(1+x^2)^{3/2}}{3/2} + C$$

Example 1

Find

$$\int x^3 \cos(x^4 + 2) dx$$

Solution

$$u = x^4 + 2 \Rightarrow du = 4x^3 dx \Rightarrow \frac{1}{4} du = x^3 dx$$

$$\frac{1}{4} \int \cos u du = \frac{1}{4} \sin u + C$$

$$= \frac{1}{4} \sin(x^4 + 2) + C$$

Example 2

Evaluate

$$\int \sqrt{2x+1} \, dx$$

Solution

$$u = 2x + 1 \Rightarrow du = 2 \, dx \Rightarrow \frac{1}{2} du = dx$$

$$\begin{aligned} \frac{1}{2} \int \sqrt{u} \, du &= \frac{1}{2} \frac{u^{3/2}}{3/2} + C \\ &= \frac{(2x+1)^{3/2}}{3} + C \end{aligned}$$

Example 3

Find

$$\int \frac{x}{\sqrt{1-4x^2}} \, dx$$

Solution

$$u = 1 - 4x^2 \Rightarrow du = -8x \, dx \Rightarrow -\frac{1}{8} du = x \, dx$$

$$\begin{aligned} -\frac{1}{8} \int \frac{du}{\sqrt{u}} &= -\frac{1}{8} \frac{u^{1/2}}{1/2} + C \\ &= -\frac{2}{8} \sqrt{u} + C \\ &= -\frac{1}{4} \sqrt{1-4x^2} + C \end{aligned}$$

Example 4

Calculate

$$\int e^{5x} dx$$

Solution

$$u = 5x \Rightarrow du = 5 dx \Rightarrow \frac{1}{5} du = dx$$

$$\begin{aligned} \frac{1}{5} \int e^u du &= \frac{1}{5} e^u + C \\ &= \frac{1}{5} e^{5x} + C \end{aligned}$$

Example 5

Calculate

$$\int \sqrt{1+x^2} x^5 dx$$

Solution

$$u = 1+x^2 \Rightarrow du = 2x dx \Rightarrow \frac{1}{2} du = x dx$$

$$x^2 = u-1 \Rightarrow x^4 = (u-1)^2$$

$$\int \sqrt{1+x^2} \cdot x^4 \cdot x dx = \frac{1}{2} \int \sqrt{u} (u-1)^2 du$$

$$= \frac{1}{2} \int \sqrt{u} (u^2 - 2u + 1) du = \frac{1}{2} \int u^{5/2} - 2u^{3/2} + u^{1/2} du$$

$$= \frac{1}{2} \left(\frac{u^{7/2}}{7/2} - 2 \frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} \right) + C$$

$$= \frac{(1+x^2)^{7/2}}{7} - \frac{2(1+x^2)^{5/2}}{5} + \frac{(1+x^2)^{3/2}}{3} + C$$

- The Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

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Example 7

Evaluate

$$\int_0^4 \sqrt{2x+1} dx$$

Solution

$$u = 2x + 1 \Rightarrow du = 2 dx \Rightarrow \frac{1}{2} du = dx$$

$$u(4) = 2(4) + 1 = 9$$

$$u(0) = 2(0) + 1 = 1$$

$$\frac{1}{2} \int_1^9 \sqrt{u} du$$

$$= \frac{1}{2} \left. \frac{u^{3/2}}{3/2} \right|_1^9$$

$$= \frac{u^{3/2}}{3} \Big|_1^9$$

$$= \frac{9^{3/2}}{3} - \frac{1^{3/2}}{3} = \frac{27}{3} - \frac{1}{3}$$

$$= 9 - \frac{1}{3} = \frac{26}{3}$$

Example 8

Evaluate

$$\int_1^2 \frac{dx}{(3-5x)^2}$$

Solution

$$u = 3 - 5x \Rightarrow du = -5 dx \Rightarrow \frac{1}{-5} du = dx$$

$$u(2) = 3 - 5(2) = -7$$

$$u(1) = 3 - 5(1) = -2$$

$$\begin{aligned} -\frac{1}{5} \int_{-2}^{-7} \frac{1}{u^2} du &= \cancel{\frac{1}{5}} \left[\frac{u^{-1}}{\cancel{-1}} \right]_{-2}^{-7} \\ &= \frac{1}{5} \left[\frac{1}{-7} - \frac{1}{-2} \right] = \frac{1}{5} \left(\frac{1}{2} - \frac{1}{7} \right) \\ &= \frac{1}{5} \left(\frac{7-2}{14} \right) = \frac{1}{14} \end{aligned}$$

Example 9

Evaluate

$$\int_1^e \frac{\ln x}{x} dx$$

Solution

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$u(e) = \ln e = 1$$

$$u(1) = \ln 1 = 0$$

$$\int_0^1 u du = \left. \frac{u^2}{2} \right|_0^1 = \frac{1}{2}$$

- Symmetry

Suppose f is continuous on $[-a, a]$.

$$\int_{-a}^a f(x) dx$$

f is odd	f is even
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$$= 0$$

$$= 2 \int_0^a f(x) dx$$

Example 10

Evaluate

$$\int_{-2}^2 (x^6 + 1) dx$$

Solution

$$f(x) = x^6 + 1$$

$$f(-x) = (-x)^6 + 1 = x^6 + 1 = f(x)$$

$$\therefore \int_{-2}^2 (x^6 + 1) dx = 2 \int_0^2 (x^6 + 1) dx$$

$$= 2 \left[\frac{x^7}{7} + x \right]_0^2$$

$$= 2 \left[\left(\frac{2^7}{7} + 2 \right) - 0 \right]$$

$$= \frac{2^8}{7} + 4$$

$$= \frac{256}{7} + \frac{28}{7} = \frac{284}{7}$$

Problems

- Evaluate the integral by making the given substitution.

(a) $\int x e^{-x^2} dx, \quad u = -x^2$

(b) $\int \sin^2 \theta \cos \theta d\theta, \quad u = \sin \theta$

- Evaluate the indefinite integral.

(a) $\int e^{-5r} dr$

(b) $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

(c) $\int e^{\cos t} \sin t dt$

$$(d) \int \frac{x}{1+x^4} dx$$

$$(e) \int \frac{1+x}{1+x^2} dx$$

- Evaluate the definite integral.

(a) $\int_1^2 \frac{e^{1/x}}{x^2} dx$

(b) $\int_0^{\pi/2} \cos x \sin(\sin x) dx$

$$(c) \int_0^1 \frac{e^y+1}{e^y+y} dy$$

$$(d) \int_{-1}^1 \frac{\tan x}{1+x^2+x^4} dx$$