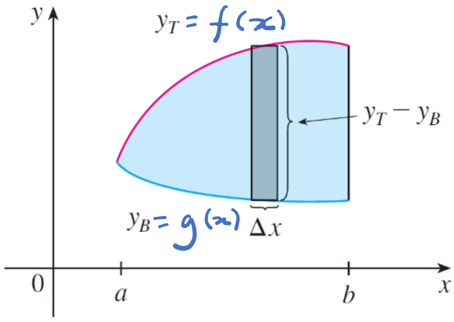
Section 6.1 – Areas Between Curves

- Areas between curves

Region Type I (Vertical): $y_T \geq y_B \quad \forall x \in [a, b]$

$$A = \int_a^b [y_T - y_B]\, dx$$

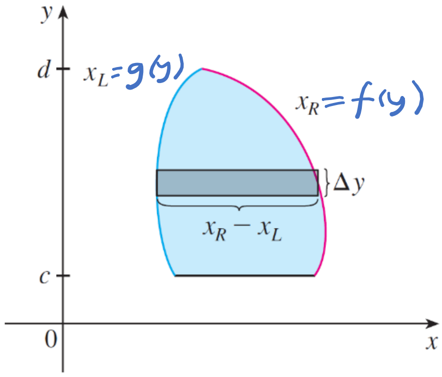
$f(x)$ $g(x)$



Region Type II (Horizontal): $x_R \geq x_L \quad \forall y \in [c, d]$

$$A = \int_c^d [x_R - x_L]\, dy$$

$f(y)$ $g(y)$



Example 1

Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Solution

$$\int_0^1 y_T - y_B \, dx$$

$$= \int_0^1 e^x - x \, dx = e^x \Big|_0^1 - \frac{x^2}{2} \Big|_0^1$$

$$= (e - 1) - \frac{1}{2}$$

$$= e - \frac{3}{2}$$

Example 2

Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Solution

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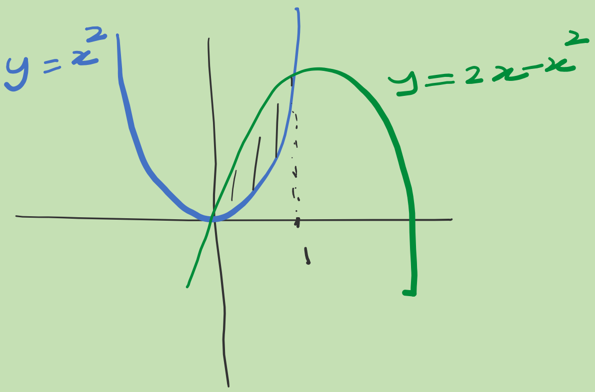
$$2x - x^2 = x^2$$

$$2x - 2x^2 = 0$$

$$2x(x-1) = 0$$

$$x = 0 \qquad x = 1$$



لمعرفة y_T و y_B

$$y_1\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

$$y_2\left(\tfrac{1}{2}\right) = 2\left(\tfrac{1}{2}\right) - \left(\tfrac{1}{2}\right)^2$$

$$= 1 - \tfrac{1}{4} = \tfrac{3}{4}$$

$$A = \int_0^1 2x - x^2 - x^2 \, dx$$

$$= \frac{\cancel{2}x^2}{\cancel{2}} - \frac{2x^3}{3} \Bigg|_0^1$$

$$= 1 - \frac{2}{3} = \frac{1}{3}$$

Example 3

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution

$$x = y + 1$$

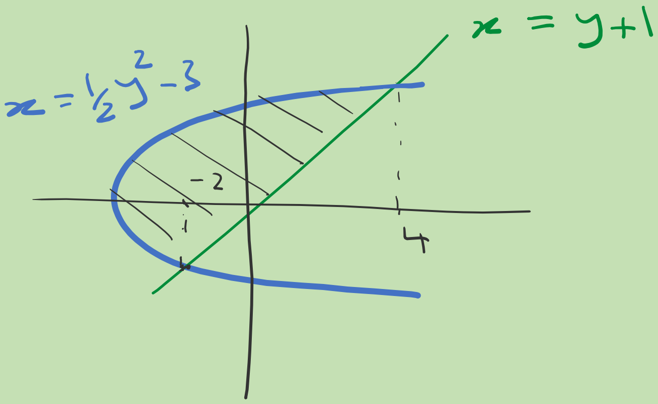
$$x = \frac{1}{2}y^2 - 3$$

$$y + 1 = \frac{1}{2}y^2 - 3 \Rightarrow \frac{1}{2}y^2 - y - 4 = 0 \quad \times 2$$

$$y^2 - 2y - 8 = 0$$

$$(y - 4)(y + 2) = 0$$

$$y = 4 \qquad\qquad y = -2$$



$$x_1(0) = 0 + 1 = 1$$

$$x_2(0) = \tfrac{1}{2}(0) - 3$$

$$= -3$$

$$A = \int_{-2}^{4} y + 1 - \left(\tfrac{1}{2}y^2 - 3\right) \, dy$$

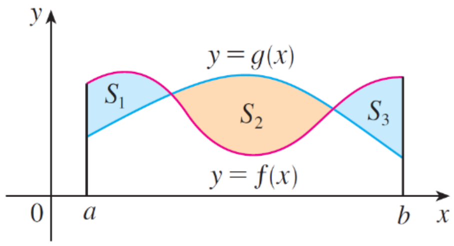
$$= \int_{-2}^{4} -\tfrac{1}{2}y^2 + y + 4 \, dy = -\tfrac{1}{2}\frac{y^3}{3} + \frac{y^2}{2} + 4y \Big|_{-2}^{4}$$

$$= -\frac{4^3}{6} + \frac{4^2}{2} + 4 \cdot 4 - \left(-\frac{(-2)^3}{6} + \frac{(-2)^2}{2} + 4 \cdot -2\right) = 18$$

- **Varying inequality**

The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\, dx$$



$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geq g(x) \\ g(x) - f(x) & \text{when } g(x) \geq f(x) \end{cases}$$

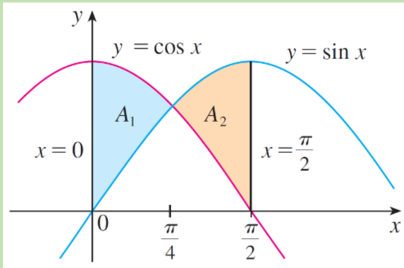
Example 4

Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

Solution

$\sin x = \cos x \qquad x \in [0, \pi/2]$

$x = \frac{\pi}{4}$



$A = A_1 + A_2$

$= \int_0^{\pi/4} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx$

$= \sin x + \cos x \Big|_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2}$

$= \Big[\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - \left(\cancel{\sin 0}^{\,0} + \cos 0\right)\Big]$

$+ \Big[\left(-\cancel{\cos\frac{\pi}{2}}^{\,0} - \sin\frac{\pi}{2}\right) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)\Big]$

$= \Big[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\Big] + \Big[-1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\Big] = 2\sqrt{2} - 2$

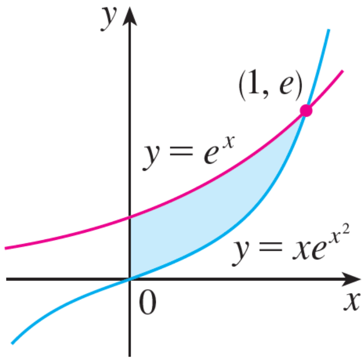
Problems

- Find the area of the shaded region.

(a) $A = \int_0^1 e^x - x e^{x^2} \, dx$

$= \int_0^1 e^x \, dx - \int_0^1 x e^{x^2} \, dx$

$= e^x \Big|_0^1 - \int_0^1 x e^{x^2} \, dx$



$u = x^2 \Rightarrow u(0) = 0, \; u(1) = 1$

$du = 2x \, dx$

$\frac{1}{2} du = x \, dx$

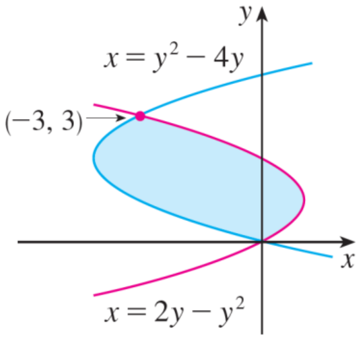
$A = e - 1 - \frac{1}{2} \int_0^1 e^u \, du$

$= e - 1 - \frac{1}{2} \left[e^u \right]_0^1$

$= e - 1 - \frac{1}{2} \left[e - 1 \right]$

$= \frac{1}{2} e - \frac{1}{2}$

(b) $$A = \int_0^3 2y - y^2 - (y^2 - 4y)\, dy$$



$$= \int_0^3 -2y^2 + 6y\, dy$$

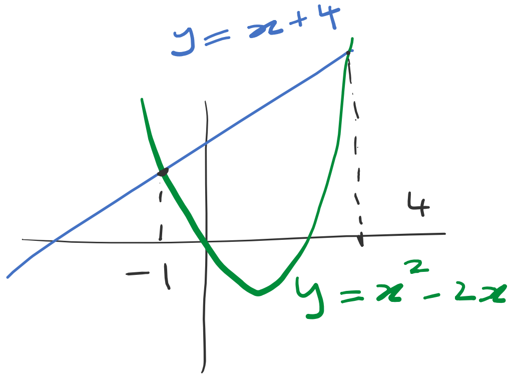
$$= -\frac{2y^3}{3} + \frac{6y^2}{2} \Big|_0^3$$

$$= -\frac{2}{3} 3^3 + 3(3^2) - 0$$

$$= -18 + 27 = 9$$

- Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Then find the area of the region.

(a) $y = x^2 - 2x, \quad y = x + 4$



$$x^2 - 2x = x + 4$$

$$x^2 - 3x - 4 = 0$$

$$(x-4)(x+1) = 0$$

$$x = 4 \qquad x = -1$$

$$A = \int_{-1}^{4} x + 4 - (x^2 - 2x)\, dx$$

$$= \int_{-1}^{4} -x^2 + 3x + 4 \, dx$$

$$= -\frac{x^3}{3} + \frac{3x^2}{2} + 4x \Big|_{-1}^{4}$$

$$= -\frac{4^3}{3} + \frac{3(4^2)}{2} + 4 \cdot 4 - \left(-\frac{(-1)^3}{3} + \frac{3(-1)^2}{2} + 4 \cdot -1\right)$$

$$= -\frac{64}{3} + 24 + 16 - \frac{1}{3} - \frac{3}{2} + 4$$

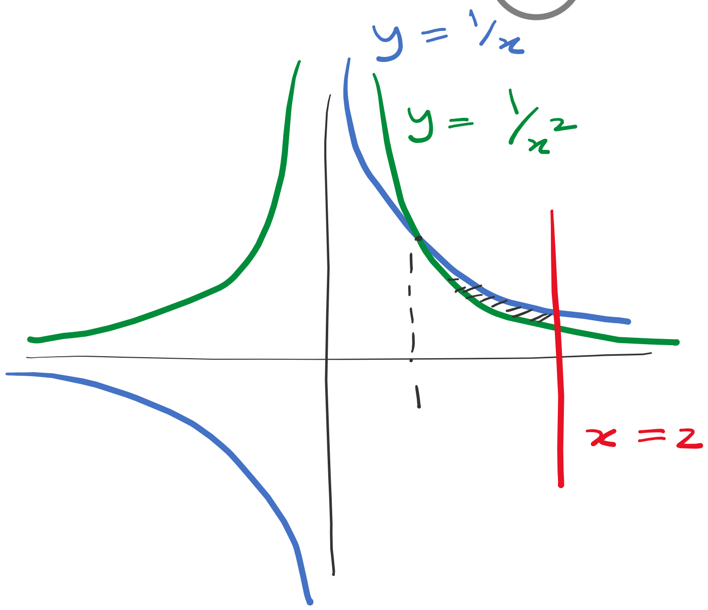
$$= \frac{125}{6}$$

(b) $y = 1/x, \quad y = 1/x^2, \quad x = 2$

$$\frac{1}{x} = \frac{1}{x^2}$$

$$\frac{x^2}{x} = 1$$

$$x = 1$$



$$A = \int_1^2 \frac{1}{x} - \frac{1}{x^2}\, dx$$

$$= \ln x - \frac{x^{-1}}{-1} \Big|_1^2$$

$$= \ln 2 + \frac{1}{2} - (\cancel{\ln 1}^{\,0} + 1)$$

$$= \ln 2 - \frac{1}{2}$$

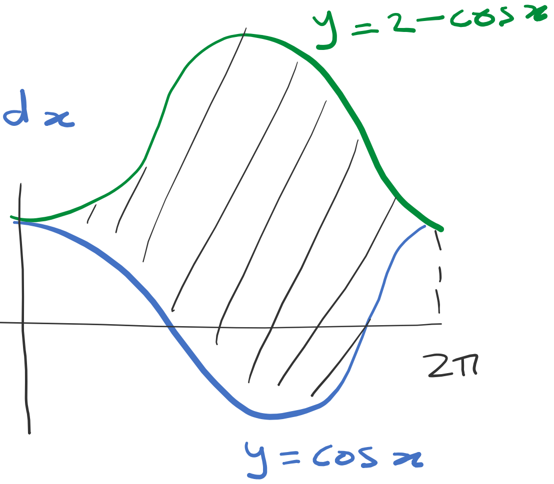
- **Sketch the region enclosed by the given curves and find its area.**

(a) $y = \cos x, \quad y = 2 - \cos x, \quad 0 \le x \le 2\pi$

$$A = \int_0^{2\pi} 2 - \cos x - \cos x \, dx$$

$$= 2x - 2\sin x \Big|_0^{2\pi}$$

$$= 4\pi$$



(b) $x = 2y^2, \quad x = 4 + y^2$

$$2y^2 = 4 + y^2$$

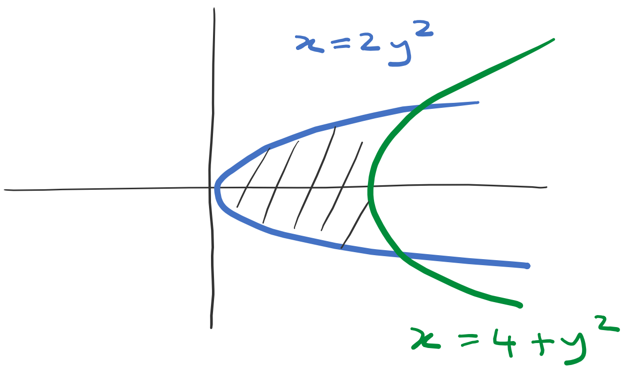
$$y^2 = 4 \Rightarrow y = \pm 2$$

$$A = \int_{-2}^{2} 4 + y^2 - 2y^2 \, dy$$

$$= 4y - \frac{y^3}{3} \Big|_{-2}^{2}$$

$$= 4(2) - \frac{2^3}{3} - \left(4(-2) - \frac{(-2)^3}{3}\right)$$

$$= 8 - \frac{8}{3} + 8 - \frac{8}{3} = 16 - \frac{16}{3} = \frac{32}{3}$$



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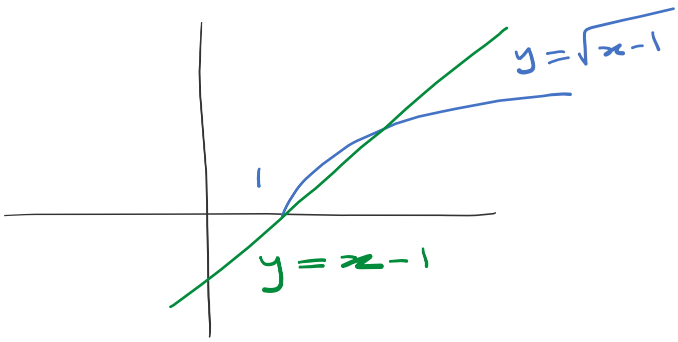
by symmetry $A = 2\int_0^2 4 - y^2 \, dy = \cdots = \frac{32}{3}$

(c) $y = \sqrt{x-1}, \quad x - y = 1$



$$y = x - 1$$

$$\sqrt{x-1} = x - 1$$

$$x - 1 = (x-1)^2$$

$$x - 1 = x^2 - 2x + 1$$

$$x^2 - 3x + 2 = 0$$

$$(x-2)(x-1) = 0$$

$$x = 2 \qquad\qquad x = 1$$

$$A = \int_1^2 \sqrt{x-1} - (x-1)\, dx$$

$$u = x - 1 \Rightarrow du = dx$$

$$u(1) = 0, \quad u(2) = 1$$

$$A = \int_0^1 u^{1/2} - u \quad du$$

$$= \frac{2}{3} u^{3/2} - \frac{u^2}{2} \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$